One of the systems of a communication satellite consists of four unreliable components each of which are necessary for successful operation of the satellite—the probabilities that a component survives the planned lifetime of the satellite (i.e., the reliabilities) are shown below:

Reliability of system

- $R_1 = 70\%$
- $R_2 = 85\%$
- $R_3 = 75\%$
- $R_4 = 68\%$

Assuming that component failures are independent,

Reliability of system

- $P\{\text{components 1 through 4 survive}\}$
- $P\{\# 1 \text{ survives}\} \times P\{\# 2 \text{ survives}\} \times P\{\# 3 \text{ survives}\} \times P\{\# 4 \text{ survives}\}$
- $= 0.70 \times 0.85 \times 0.75 \times 0.88 = 39.27\%$

This is an unacceptably low system reliability, and so redundant units of one or more components will be used in the design.

The reliability of a component may be increased by including redundant units!

Reliability of component #1

- $P\{\text{at least one unit survives}\}$
- $P\{\#1 \text{ survives}\} \times P\{\#2 \text{ survives}\} \times P\{\#3 \text{ survives}\} \times P\{\#4 \text{ survives}\}$
- $= 1 - P\{\#1 \text{ fails}\} 
  = 1 - 0.70 \times 0.85 \times 0.75 \times 0.88 = 91\%$

This assumes what is referred to as “hot standby”, i.e., a standby unit may fail even before it is put into service!
By using redundant units of each component, the system reliability can be dramatically increased—for example:

\[
\text{System Reliability} = \left[1 - (0.30)^2\right] \times \left[1 - (0.15)^2\right] \times \left[1 - (0.25)^2\right] \times 0.88
\]

\[
= 0.91 \times 0.9775 \times 0.984375 \times 0.88 = 77.0551\%
\]

The problem faced by the designer is to maximize the system reliability, subject to a restriction on the total weight of the system.

<table>
<thead>
<tr>
<th>Component</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight (kg)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Total weight must not exceed 12 kg.

*(Total weight of one unit of each component is 7 kg, leaving 5 kg for redundant units.)*

The Reliability (%) vs. # redundant units table is as follows:

<table>
<thead>
<tr>
<th>Component</th>
<th>1 unit</th>
<th>2 units</th>
<th>3 units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>91</td>
<td>97.3</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>97.75</td>
<td>99.6625</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>93.75</td>
<td>98.4375</td>
</tr>
<tr>
<td>4</td>
<td>88</td>
<td>98.56</td>
<td>99.8272</td>
</tr>
</tbody>
</table>

*We will assume that no more than three units of any component will be included!*
We impose a sequential decision-making structure on the problem by supposing that we consider the components one at a time, deciding how many units to include based upon the available weight capacity.

Arbitrarily we will use a “backward” order in what follows! That is, imagine that we first consider how many units of component #4 are to be included when we begin with 12 kg of available capacity, while component #1 is the last to be considered.

Optimal Value Function

\[ f_n(s_n) = \text{maximum reliability of the subsystem consisting of devices } n, n-1, \ldots, 1, \text{ if } s_n \text{ kg of available capacity remains to be allocated.} \]

Recursive definition of function

\[
\begin{align*}
  f_s(s_n) &= \text{maximum} \left\{ \left(1 - p_n^{s_n}\right) \times f_{n-1}(s_{n-1} - w_n x_n) \right\} \\
  f_0(s_0) &= \begin{cases} 
  1 & \text{if } s_0 \geq 0 \\
  0 & \text{otherwise}
  \end{cases}
\end{align*}
\]

APL function definition

```
v z+F N:t
[1]   a ≡ Optimal redundancy to maximize reliability
[2]   n ≡ N-0
[4]   :if N=0
[5]   z*{(1*p[1]),-BIG
[7]   a Recursive definition of optimal value function
[8]   z*Maximize {(1*p[1])+.-|1-R[N]||x|)}*{(F N-l)[TRANSITION s*,-N[N]*x
[9]   :endif
```

Component #1: reliability = 70%, weight = 1 kg.

<table>
<thead>
<tr>
<th>Stage 1</th>
<th>s \ x:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0.7000</td>
<td>99.9999</td>
<td>99.9999</td>
<td>0.7000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.7000</td>
<td>0.9100</td>
<td>99.9999</td>
<td>0.9100</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.7000</td>
<td>0.9100</td>
<td>0.9730</td>
<td>0.9730</td>
</tr>
</tbody>
</table>

etc.
Component #2: reliability = 80%, weight = 2 kg.

Stage 2

<table>
<thead>
<tr>
<th>s</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.5600</td>
<td>99.9999</td>
<td>99.9999</td>
<td>0.5600</td>
</tr>
<tr>
<td>4</td>
<td>0.7280</td>
<td>99.9999</td>
<td>99.9999</td>
<td>0.7280</td>
</tr>
<tr>
<td>5</td>
<td>0.7784</td>
<td>0.6720</td>
<td>99.9999</td>
<td>0.7784</td>
</tr>
<tr>
<td>6</td>
<td>0.7784</td>
<td>0.8736</td>
<td>99.9999</td>
<td>0.8736</td>
</tr>
<tr>
<td>7</td>
<td>0.7784</td>
<td>0.9341</td>
<td>0.6944</td>
<td>0.9341</td>
</tr>
<tr>
<td>8</td>
<td>0.7784</td>
<td>0.9341</td>
<td>0.9027</td>
<td>0.9341</td>
</tr>
</tbody>
</table>

For example, suppose that we have 6 kg of capacity remaining, i.e., $s_2 = 6$, and we choose to include 2 units of component #2. Then we obtain 97.75% reliability of subsystem #2 and arrive at stage 1 (component #1) with $6 - 2 \times 2 = 2$ kg of capacity remaining, so that we can achieve 91% reliability ($f_1(2) = 0.91$) in subsystem #1. Hence, the subsystem of components 1&2 will have reliability $0.9775 \times 0.91 = 0.8736$.

Component #3: reliability = 75%, weight = 1 kg.

Stage 3

<table>
<thead>
<tr>
<th>s</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.4200</td>
<td>99.9999</td>
<td>99.9999</td>
<td>0.4200</td>
</tr>
<tr>
<td>5</td>
<td>0.5460</td>
<td>0.5250</td>
<td>99.9999</td>
<td>0.5460</td>
</tr>
<tr>
<td>6</td>
<td>0.5838</td>
<td>0.6825</td>
<td>0.5513</td>
<td>0.6825</td>
</tr>
<tr>
<td>7</td>
<td>0.6552</td>
<td>0.7298</td>
<td>0.7166</td>
<td>0.7298</td>
</tr>
<tr>
<td>8</td>
<td>0.7006</td>
<td>0.8190</td>
<td>0.7662</td>
<td>0.8190</td>
</tr>
<tr>
<td>9</td>
<td>0.7006</td>
<td>0.8757</td>
<td>0.8600</td>
<td>0.8757</td>
</tr>
</tbody>
</table>

Stage 4

<table>
<thead>
<tr>
<th>s</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.3696</td>
<td>99.9999</td>
<td>99.9999</td>
<td>0.3696</td>
</tr>
<tr>
<td>8</td>
<td>0.4805</td>
<td>99.9999</td>
<td>99.9999</td>
<td>0.4805</td>
</tr>
<tr>
<td>9</td>
<td>0.6006</td>
<td>99.9999</td>
<td>99.9999</td>
<td>0.6006</td>
</tr>
<tr>
<td>10</td>
<td>0.6422</td>
<td>0.4140</td>
<td>99.9999</td>
<td>0.6422</td>
</tr>
<tr>
<td>11</td>
<td>0.7207</td>
<td>0.5381</td>
<td>99.9999</td>
<td>0.7207</td>
</tr>
<tr>
<td>12</td>
<td>0.7706</td>
<td>0.6727</td>
<td>99.9999</td>
<td>0.7706</td>
</tr>
</tbody>
</table>

**Summary of computations**

Only the last row of this table need be computed to find the optimal reliability with 12 kg of capacity!
The maximum reliability, then, given a 12 kg weight restriction, is \( f_4(12) = 77.06 \% \).

By a “forward pass” through the tables, we can determine the optimal design:

<table>
<thead>
<tr>
<th>stage</th>
<th>state</th>
<th>decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>cap 12</td>
<td>1 units</td>
</tr>
<tr>
<td>3</td>
<td>cap 9</td>
<td>2 units</td>
</tr>
<tr>
<td>2</td>
<td>cap 7</td>
<td>2 units</td>
</tr>
<tr>
<td>1</td>
<td>cap 3</td>
<td>3 units</td>
</tr>
<tr>
<td>0</td>
<td>cap 0</td>
<td></td>
</tr>
</tbody>
</table>

That is, the optimal design includes 1 of component #4, 2 each of components #2 & #3, and 3 of component #1.

• What reduction in reliability would occur if the weight restriction were 11 kg rather than 12?

• What is the optimal design with a weight restriction of 11 kg?

**Integer Programming Model**

Define binary decision variables:

\[ X_{in} = 1 \text{ if } n \text{ units of component } i \text{ are included in the system} \]

\[ X_{in} = 0 \text{ otherwise} \]

Notation:

<table>
<thead>
<tr>
<th>Component</th>
<th>( R_{i1} )</th>
<th>( R_{i2} )</th>
<th>( R_{i3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.70</td>
<td>0.91</td>
<td>0.973</td>
</tr>
<tr>
<td>2</td>
<td>0.80</td>
<td>0.9775</td>
<td>0.996625</td>
</tr>
<tr>
<td>3</td>
<td>0.75</td>
<td>0.9375</td>
<td>0.984375</td>
</tr>
<tr>
<td>4</td>
<td>0.88</td>
<td>0.9856</td>
<td>0.998272</td>
</tr>
</tbody>
</table>
**Objective:**
In order to linearize the objective, we will instead maximize the **logarithm of the reliability:**

\[
\text{Maximize} \quad \sum_{i=1}^{4} \sum_{n=1}^{3} (\ln R_{in}) X_{in} \]

subject to

\[
\sum_{i=1}^{4} \sum_{n=1}^{3} (W_{in}) X_{in} \leq W_{\text{max}}
\]

\[
\sum_{n=1}^{3} X_{in} = 1 \quad \forall i = 1, 2, 3, 4
\]

\[
X_{in} \in \{0, 1\} \quad \forall i \& n
\]

**LINGO model:**

\[
\text{SETS:}
\]
\[
\text{COMPONENT / A B C D/;}
\]
\[
\text{WEIGHT;}
\]
\[
\text{UNITS / 1..3/;}
\]
\[
\text{LOG (COMPONENT,UNITS): LNR, X;}
\]
\[
\text{ENDSETS}
\]
\[
\text{DATA:}
\]
\[
\text{WEIGHT = 1 2 1 3;}
\]
\[
\text{WMAX = 12;}
\]
\[
\text{LNR = -0.35667 -0.094311 -0.027371 -0.22314 -0.040822 -0.0080322 -0.28768 -0.064539 -0.015748 -0.12783 -0.014505 -0.0017295;}
\]
\[
\text{ENDDATA}
\]
\[
\text{MAX = @SUM ( COMPONENT(I): @SUM(UNITS(N):LNR(I,N)*X(I,N))) ;}
\]
\[
\text{@SUM( COMPONENT(I): @SUM(UNITS(N): WEIGHT(I)*N*X(I,N))) <= WMAX;}
\]
\[
\text{@FOR (COMPONENT(I):)
\text{ @SUM(UNITS(N): X(I,N))=1; )};
\]
\[
\text{@FOR (COMPONENT(I):)
\text{ @BIN (X(I,N)) ) );}
\]

**LINGO model:**

\[
\text{MAX}
\]
\[
-0.35667 \text{ X( A, 1)} - 0.094311 \text{ X( A, 2)} - 0.027371 \text{ X( A, 3)}
\]
\[
-0.22314 \text{ X( B, 1)} - 0.040822 \text{ X( B, 2)} - 0.0080322 \text{ X( B, 3)}
\]
\[
-0.28768 \text{ X( C, 1)} - 0.064539 \text{ X( C, 2)} - 0.015748 \text{ X( C, 3)}
\]
\[
-0.12783 \text{ X( D, 1)} - 0.014505 \text{ X( D, 2)} - 0.0017295 \text{ X( D, 3)}
\]
\[
\text{SUBJECT TO}
\]
\[
2] \text{ X( A, 1)} + 2 \text{ X( A, 2)} + 3 \text{ X( A, 3)} + 2 \text{ X( B, 1)} + 4 \text{ X( B, 2)} + 6 \text{ X( B, 3)} + 2 \text{ X( C, 1)} + 2 \text{ X( C, 2)} + 3 \text{ X( C, 3)} + 3 \text{ X( D, 1)} + 6 \text{ X( D, 2)} + 9 \text{ X( D, 3)} <= 12
\]
\[
3] \text{ X( A, 1)} + \text{ X( A, 2)} + \text{ X( A, 3)} = 1
\]
\[
4] \text{ X( B, 1)} + \text{ X( B, 2)} + \text{ X( B, 3)} = 1
\]
\[
5] \text{ X( C, 1)} + \text{ X( C, 2)} + \text{ X( C, 3)} = 1
\]
\[
6] \text{ X( D, 1)} + \text{ X( D, 2)} + \text{ X( D, 3)} = 1
\]
\[
\text{END}
\]
\[
\text{INTE 12}
\]
**Optimal Solution:**

Objective value:  - 0.2605620

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>X(A, 3)</td>
<td>1.000000</td>
<td>0.2737100E-01</td>
</tr>
<tr>
<td>X(B, 2)</td>
<td>1.000000</td>
<td>0.4082200E-01</td>
</tr>
<tr>
<td>X(C, 2)</td>
<td>1.000000</td>
<td>0.6453900E-01</td>
</tr>
<tr>
<td>X(D, 1)</td>
<td>1.000000</td>
<td>0.1278300</td>
</tr>
</tbody>
</table>

*Note that* \( \exp(-0.2605620) = 0.77062 \) which is in agreement with the dynamic programming solution.