Deterministic Equipment Replacement Problem

Suppose that a new car costs $20,000 and that the annual operating cost and resale value for a car is determined only by its age:

Age i (years)	Operating cost C_i next year ($)	Resale value V_i ($)
0	400	
1	600	16,000
2	900	13,000
3	1300	9,000
4	1800	6,000
5	2400	4,000
6	3100	2,000
7	3900	1,000
8	4800	1,000
9	5800	1,000
10	7000	000

What is the optimal replacement strategy for the next 10 years, given that you begin with a new car, and at the end of the ten years, you receive the trade-in value for your current car?

DP Model #1

Define stages corresponding to the years, with

stage n = # years remaining in the planning period

state s = age of car at current stage

decision x = 0 if "keep", 1 if "replace"

$f_n(s)$= minimum cost of last n years if the age of the car is currently s

Transition: If the age at stage n is s, then the age at stage $n-1$ is

$$\begin{cases} s+1 & \text{if } x=0 \text{ ("keep")} \\ 1 & \text{if } x=1 \text{ ("replace")} \end{cases}$$

Recursion:

$$f_n(s) = \min \left\{ C_s + f_{n-1}(s+1),\quad p - V_s + C_0 + f_{n-1}(1) \right\}$$

where the first & second elements in the minimand correspond to the decisions "keep" & "replace", respectively.

APL Function Definition

```
    ∇ z←F N;operating_cost;OC;PC;TIV
   ⍝
   ⍝ Value Function for deterministic auto replacement
   ⍝ where s = age, x = [0, 1] = [keep, replace]
   ⍝
   :if N=0    ⍝ Terminal conditions
     z←(-(⍴s)⍴V),BIG
     ⍝  Penalty appended to prevent infeasible states
   :else
     ⍝  recursion
     OC←C[s⍳s∘.×~x]           ⍝ Operating cost
     PC←((⍴s)⍴PP)∘.×x         ⍝ Purchase cost
     TIV←V∘.×x                ⍝ Trade-in value
     z←MIN (OC+PC-TIV)+(F N-1)[TRANSITION 1+s∘.×~x]
   :endif
    ∇
```

---Stage 1---

s \ x:	0	1	Minimum
1	¯12400	¯11600	¯12400
2	¯8100	¯8600	¯8600
3	¯4700	¯4600	¯4700
4	¯2200	¯1600	¯2200
5	400	400	400
6	2100	2400	2100
7	2900	3400	2900
8	3800	3400	3400
9	5800	3400	3400

---Stage 2---

s \ x:	0	1	Minimum
1	¯8000	¯8000	¯8000
2	¯3800	¯5000	¯5000
3	¯900	¯1000	¯1000
4	2200	2000	2000
5	4500	4000	4000
6	6000	6000	6000
7	7300	7000	7000
8	8200	7000	7000

---Stage 3---

s \ x:	0	1	Minimum
1	¯4400	¯3600	¯4400
2	¯100	¯600	¯600
3	3300	3400	3300
4	5800	6400	5800
5	8400	8400	8400
6	10100	10400	10100
7	10900	11400	10900

---Stage 4---

s \ x:	0	1	Minimum
1	0	0	0
2	4200	3000	3000
3	7100	7000	7000
4	10200	10000	10000
5	12500	12000	12000
6	14000	14000	14000

---Stage 5---

s \ x:	0	1	Minimum
1	3600	4400	3600
2	7900	7400	7400
3	11300	11400	11300
4	13800	14400	13800
5	16400	16400	16400

---Stage 6---

s \ x:	0	1	Minimum
1	8000	8000	8000
2	12200	11000	11000
3	15100	15000	15000
4	18200	18000	18000

---Stage 7---

s \ x:	0	1	Minimum
1	11600	12400	11600
2	15900	15400	15400
3	19300	19400	19300

---Stage 8---

s \ x:	0	1	Minimum
1	16000	16000	16000
2	20200	19000	19000

---Stage 9---

s \ x:	0	1	Minimum
1	19600	20400	19600

---Stage 10---

s \ x:	0	1	Minimum
0	20000	22000	20000

*** Optimal value is 20000.00 ***

stage	state	decision
10	0	Keep
9	1	Keep
8	2	Replace
7	1	Keep
6	2	Replace
5	1	Keep
4	2	Replace
3	1	Keep
2	2	Replace
1	1	Keep
0	2	

DP Model #2

Define the optimal value function

$g(n)$= minimum cost of last n years, if at the beginning of that time one has a new car.

We wish to determine $g(10)$.

Decision variable: y = # of years to keep car before replacement

Recursion:

$$g(n) = \min_{1\le y\le n} \left\{ \sum_{i=0}^{y-1} C_i + \left(p - V_y\right) + g(n-y) \right\}, \quad n = 1, 2, \ldots$$

$$g(0) = 0$$

APL Code for Recursive Computation

```
Solve N;OC;RC;TV;z;m
⍝
⍝    Deterministic equipment replacement
⍝    Assumes machine is NOT replaced at final stage!
g←y←,0

:for n :in ⍳N
  m←Max_Age⌊n
  OC←+\m↑C                ⍝ Operating cost
  RC←(((m-1)⍴PP),0)-m↑V                 ⍝ Replacement cost
  z←OC+RC+m↑⌽g            ⍝ Total
  g←g,⌊/z                          ⍝ Find minimum
  y←y,z⍳⌊/z                        ⍝ Find minimizer
:endfor
```


Details of Solution

n = 1

y	OC	RC	g[°]	Total
1	400	¯16000	0	¯15600

Minimum is ¯15600 at y = 1

n = 2

y	OC	RC	g[°]	Total
1	400	4000	¯15600	¯11200
2	1000	¯13000	0	¯12000

Minimum is ¯12000 at y = 2

n = 3

y	OC	RC	g[°]	Total
1	400	4000	¯12000	¯7600
2	1000	7000	¯15600	¯7600
3	1900	¯9000	0	¯7100

Minimum is ¯7600 at y = 1

n = 4

y	OC	RC	g[°]	Total
1	400	4000	¯7600	¯3200
2	1000	7000	¯12000	¯4000
3	1900	11000	¯15600	¯2700
4	3200	¯6000	0	¯2800

Minimum is ¯4000 at y = 2

n = 5

y	OC	RC	g[°]	Total
1	400	4000	¯4000	400
2	1000	7000	¯7600	400
3	1900	11000	¯12000	900
4	3200	14000	¯15600	1600
5	5000	¯4000	0	1000

Minimum is 400 at y = 1

n = 6

y	OC	RC	g[°]	Total
1	400	4000	400	4800
2	1000	7000	¯4000	4000
3	1900	11000	¯7600	5300
4	3200	14000	¯12000	5200
5	5000	16000	¯15600	5400
6	7400	¯2000	0	5400

Minimum is 4000 at y = 2

n = 7

y	OC	RC	g[°]	Total
1	400	4000	4000	8400
2	1000	7000	400	8400
3	1900	11000	¯4000	8900
4	3200	14000	¯7600	9600
5	5000	16000	¯12000	9000
6	7400	18000	¯15600	9800
7	10500	¯1000	0	9500

Minimum is 8400 at y = 1

n = 8

y	OC	RC	g[°]	Total
1	400	4000	8400	12800
2	1000	7000	4000	12000
3	1900	11000	400	13300
4	3200	14000	¯4000	13200
5	5000	16000	¯7600	13400
6	7400	18000	¯12000	13400
7	10500	19000	¯15600	13900
8	14400	¯1000	0	13400

Minimum is 12000 at y = 2

n = 9

y	OC	RC	g[°]	Total
1	400	4000	12000	16400
2	1000	7000	8400	16400
3	1900	11000	4000	16900
4	3200	14000	400	17600
5	5000	16000	¯4000	17000
6	7400	18000	¯7600	17800
7	10500	19000	¯12000	17500
8	14400	19000	¯15600	17800
9	19200	¯1000	0	18200

Minimum is 16400 at y = 1

n = 10

y	OC	RC	g[°]	Total
1	400	4000	16400	20800
2	1000	7000	12000	20000
3	1900	11000	8400	21300
4	3200	14000	4000	21200
5	5000	16000	400	21400
6	7400	18000	¯4000	21400
7	10500	19000	¯7600	21900
8	14400	19000	¯12000	21400
9	19200	19000	¯15600	22600
10	25000	0	0	25000

Minimum is 20000 at y = 2

Summary of Results

n	g(n)	y(n)
0	0	0
1	¯15600	1
2	¯12000	2
3	¯7600	1
4	¯4000	2
5	400	1
6	4000	2
7	8400	1
8	12000	2
9	16400	1
10	20000	2

Verify that this is the same replacement plan as that of the first model!

The second DP model, based upon a renewal process, requires significantly less computation than the first.

How can the stochastic nature of the real process be incorporated into each DP model?