The property of "CONVEXITY" of sets and of functions is central to most approaches to nonlinear programming.

**Convex Set**

A set $S$ is **convex** if for every pair of elements $x$ and $y$ in $S$, the line segment joining $x$ and $y$ also lies in $S$.

The **line segment** between $x$ and $y$ is given by

$$\lambda y + (1-\lambda)x = x + \lambda(y-x) \quad \text{for } \lambda \in [0,1]$$

where

$$\lambda = 0 \Rightarrow \lambda y + (1-\lambda)x = x$$

$$\lambda = 1 \Rightarrow \lambda y + (1-\lambda)x = y$$

$$\lambda = \frac{1}{2} \Rightarrow \lambda y + (1-\lambda)x = \frac{1}{2}(x+y) \quad \text{(midpt of segment)}$$

etc.

**Convex Combination**

If $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ are vectors in $\mathbb{R}^n$, and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are nonnegative numbers whose sum is 1, i.e., $\sum_{i=1}^{k} \lambda_i = 1$ then

$$\lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_k x^{(k)}$$

is a convex combination (weighted average) of $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$.

In particular, a point on a line segment is a convex combination of the endpoints of the segment.

The **convex hull** of a set of points is the set of all convex combinations of those points.
Theorem: If $x^{(1)}, x^{(2)}, \ldots x^{(k)} \in S$ where $S$ is a convex set, then every convex combination of the points $x^{(1)}, x^{(2)}, \ldots x^{(k)}$ is an element of $S$.

That is, if $S$ is convex then $S$ equals its convex hull.

Convex Function: A function $f(x)$ is convex if:

$$f(\lambda x^{(1)} + (1-\lambda) x^{(2)}) \leq \lambda f(x^{(1)}) + (1-\lambda) f(x^{(2)}) \quad \forall \lambda \in [0,1]$$

For example, $f$ evaluated at the midpoint of two points is less than the average of the function values at the two points.

Definition: $f(x)$

Convex Function: A differentiable function $f(x)$ is convex if:

The tangent line (hyperplane) to the graph lies on or below the graph:

Convex Function: A differentiable function $f(x)$ is convex if:

$$f(x^{(1)}) + f'(x^{(1)})(x^{(2)} - x^{(1)}) \leq f(x^{(2)})$$

The epigraph of a function is the set

$$\text{epi}(f) = \{ (y,x) \mid y \geq f(x) \}$$

For any real value $c$, the Level Set of the function $f$ is the set

$$L_c^f = \{ x \mid f(x) \leq c \}$$

Property of convex function
If \( f \) is a convex function, then \( L_f \) is convex.

However, the convexity of the level sets does NOT imply convexity of the function.

If all the level sets of a function are convex, then the function is \emph{quasi-convex}.

\[ \text{Examples} \]

\begin{align*}
\text{convex functions} & & \text{concave functions}
\end{align*}

\[ \text{Examples} \]

\[ \boxed{\begin{align*}
\text{Concave Function} & : \text{A function } f \text{ is } \textit{concave} \text{ if} \\
& \bullet \text{ its negative, } (-f), \text{ is convex} \\
& \bullet \text{ a chord between } 2 \text{ points on the graph lies on or below the graph} \\
& \bullet \text{ a tangent line (hyperplane) to the graph lies on or above the graph} \\
& \bullet \text{ the hypergraph } \{ (y,x) \mid y \leq f(x) \} \text{ is convex}
\end{align*}} \]

\[ \text{Examples} \]

\[ \boxed{\begin{align*}
\text{A linear function is } \textit{both} \text{ convex and concave!}
\end{align*}} \]

\[ \text{A function may be neither convex nor concave:} \]

\[ \boxed{\begin{align*}
\end{align*}} \]