**Center Problems in a Network**

Define the function $\sigma(x) = \max_{j \in N} d(x,j)$ where $d(x,j)$ is the shortest path from $x$ to node $j$, i.e., the distance from $x$ to the farthest node of the network.

Suppose $x \in \text{edge } [P,Q]$.

$$d(x,j) = \text{shortest path from } x \text{ to node } j$$

$$= \min \{ d(X,P) + d(P,j) \text{ and } d(X,Q) + d(Q,j) \}$$

i.e., the lower envelope of two linear functions.

For $x \in \text{edge } [P,Q]$, $\sigma(x)$ is the upper envelope of the functions $d(X,j)$ for $j \in N$.

The **Vertex Center** is the point $x \in N$ which solves

$$\min_x \sigma(x)$$

i.e., the point which solves the minimax problem

$$\min_x \left\{ \max_{j \in N} d(x,j) \right\}$$

The **Edge Center** of an edge $[J,K]$ is the point $z$ on edge $[J,K]$ which solves

$$\min_{x \in [J,K]} \sigma(x)$$

The **Absolute Center** of a network is the point $z$ (a node or a point on an edge) which solves

$$\min_{x \in G} \sigma(x)$$

where $G = N \cup A$ is the set of nodes and points on edges in the edge set $A$.

**Example**

Where should a fire station be located so as to minimize the distance to the farthest village?

$$d(x,j) = \text{shortest path from point } x \text{ (on the network) to village } J, J \in N = \{A,B,C,D,E\}$$

Minimize $\max_x \{ d(x,j) \}$.
Consider $d(x, W)$ for points $x$ on the edge $(A, B)$

$d(x, A)$ is monotonically increasing (slope = 1) as $x$ moves from $A$ to $B$, while $d(x, B)$ is monotonically decreasing (slope = -1).

$d(x, C) = 3$ at $x = A$, and increases (slope = 1) as $x$ moves toward $B$.
At the point $x$ where $d(x, A) + 2 = d(x, B) + 2$,
the function begins to decrease (slope = -1).

$z = 1.5$
$z = 1.5 

$\sigma(x) = \max_{j \in N} d(x, j)$
$\sigma(x)$ is the upper envelope of the family of functions $d(x, j), j \in N$.

The point which minimizes the function $\sigma$ on $[A, B]$ lies 0.5 miles from $A$.

The absolute center may be found by computing each edge center, and selecting the best.

$\sigma(x) = \max_{j \in N} d(x, j)$

$\sigma(x) = \max_{j \in N} d(x, j)$

$\sigma(x) = \max_{j \in N} d(x, j)$
\[ \sigma(x) = \max_{j \in N} \delta(x,j) \]

Searching some edges for their centers may be avoided by using the lower bound provided by

**Theorem**: Let \( X_{pq} \) be the edge center of \([P,Q] \).

Then

\[ \sigma(X_{pq}) \geq \frac{\sigma(P) + \sigma(Q) - d(P,Q)}{2} \]

If this lower bound exceeds \( \sigma \) at the vertex center of the network, then the absolute center cannot lie on this edge.

**Proof**: \( d(X_{pq}) \leq \sigma(X) \quad \forall \ j \)

\[ d(P,j) \leq d(P,X) + d(X,j) \]

\[ d(P,j) \leq d(P,X) + \sigma(X) \]

But \( \sigma(X) = \max_i \{ d(P,j) \} \),

\[ \Rightarrow \sigma(X) \leq d(P,X) + \sigma(X) \]

Likewise, \( \sigma(Q) \leq d(Q,X) + \sigma(X) \)

Sum these two inequalities:

\[ \sigma(P) + \sigma(Q) \geq 2 \sigma(X) + d(P,X) + d(Q,X) \]

\[ \Rightarrow \sigma(X) \geq \frac{\sigma(P) + \sigma(Q) - d(P,Q)}{2} \]

**Example**: Using the lower bound would have eliminated 3 edges from consideration.

The edge centers needed to be found only for \([A,B] \) & \([C,D] \)

**Example**: Shortest Path Lengths
Vertex Center of Network

<Which minimizes the maximum distance to farthest nodes>

Vertex center of the network is at node 6
where maximum distance (to node 10) is 57

The function \( \sigma \) on edge [6,7]

Monotonically increasing distance functions: \( d(x,k) \) where

\[
\begin{align*}
    &k = 1,2,3,4,5,6 \\
    &d(1,k) = 13, 9, 6, 4, 3, 2 \\
    &d(2,k) = 14, 9, 6, 4, 3, 2 \\
    &d(3,k) = 15, 9, 6, 4, 3, 2 \\
    &d(4,k) = 16, 9, 6, 4, 3, 2 \\
    &d(5,k) = 17, 9, 6, 4, 3, 2 \\
    &d(6,k) = 18, 9, 6, 4, 3, 2 \\
\end{align*}
\]

Monotonically decreasing distance functions: \( d(x,k) \) where

\[
\begin{align*}
    &k = 1,2,3,4,5,6 \\
    &d(1,k) = 18, 22, 26, 30, 34, 38 \\
    &d(2,k) = 17, 21, 25, 29, 33, 37 \\
    &d(3,k) = 16, 20, 24, 28, 32, 36 \\
    &d(4,k) = 15, 19, 23, 27, 31, 35 \\
    &d(5,k) = 14, 18, 22, 26, 30, 34 \\
    &d(6,k) = 13, 17, 21, 25, 29, 33 \\
\end{align*}
\]

Distance functions which increase to a peak at a point
A units from 1, then decrease: \( d(x,k) \) where

\[
\begin{align*}
    k &= 9,10 \\
    d(1,k) &= 23, 77 \\
    d(2,k) &= 50, 88 \\
    \Delta &= 15.3 \\
\end{align*}
\]

The function \( \sigma \) on edge [6,9]

Monotonically increasing distance functions: \( d(x,k) \) where

\[
\begin{align*}
    &k = 1,2,3,4,5,6 \\
    &d(1,k) = 13, 9, 6, 4, 3, 2 \\
    &d(2,k) = 14, 9, 6, 4, 3, 2 \\
    &d(3,k) = 15, 9, 6, 4, 3, 2 \\
    &d(4,k) = 16, 9, 6, 4, 3, 2 \\
    &d(5,k) = 17, 9, 6, 4, 3, 2 \\
    &d(6,k) = 18, 9, 6, 4, 3, 2 \\
\end{align*}
\]

Monotonically decreasing distance functions: \( d(x,k) \) where

\[
\begin{align*}
    &k = 1,2,3,4,5,6 \\
    &d(1,k) = 18, 22, 26, 30, 34, 38 \\
    &d(2,k) = 17, 21, 25, 29, 33, 37 \\
    &d(3,k) = 16, 20, 24, 28, 32, 36 \\
    &d(4,k) = 15, 19, 23, 27, 31, 35 \\
    &d(5,k) = 14, 18, 22, 26, 30, 34 \\
    &d(6,k) = 13, 17, 21, 25, 29, 33 \\
\end{align*}
\]

Distance functions which increase to a peak at a point
A units from 1, then decrease: \( d(x,k) \) where

\[
\begin{align*}
    k &= 7,8,9,10,11,12 \\
    d(1,k) &= 13, 17, 21, 25, 29, 33 \\
    d(2,k) &= 14, 18, 22, 26, 30, 34 \\
    \Delta &= 28.5, 28.5, 1.5, 1.5, 1.5, 1.5 \\
\end{align*}
\]

The function \( \sigma \) on edge [6,11]

Monotonically increasing distance functions: \( d(x,k) \) where

\[
\begin{align*}
    &k = 1,2,3,4,5,6 \\
    &d(1,k) = 13, 9, 6, 4, 3, 2 \\
    &d(2,k) = 14, 9, 6, 4, 3, 2 \\
    &d(3,k) = 15, 9, 6, 4, 3, 2 \\
    &d(4,k) = 16, 9, 6, 4, 3, 2 \\
    &d(5,k) = 17, 9, 6, 4, 3, 2 \\
    &d(6,k) = 18, 9, 6, 4, 3, 2 \\
\end{align*}
\]

Monotonically decreasing distance functions: \( d(x,k) \) where

\[
\begin{align*}
    &k = 1,2,3,4,5,6 \\
    &d(1,k) = 18, 22, 26, 30, 34, 38 \\
    &d(2,k) = 17, 21, 25, 29, 33, 37 \\
    &d(3,k) = 16, 20, 24, 28, 32, 36 \\
    &d(4,k) = 15, 19, 23, 27, 31, 35 \\
    &d(5,k) = 14, 18, 22, 26, 30, 34 \\
    &d(6,k) = 13, 17, 21, 25, 29, 33 \\
\end{align*}
\]

Distance functions which increase to a peak at a point
A units from 1, then decrease: \( d(x,k) \) where

\[
\begin{align*}
    k &= 9,10 \\
    d(1,k) &= 23, 77 \\
    d(2,k) &= 50, 88 \\
    \Delta &= 16.5, 12 \\
\end{align*}
\]
The function \( \sigma \) on edge \([6,12]\)

Monotonically increasing distance functions: \( d(x,k) \) where
- \( d(6,12) = 52 \) 59 61 53 50 15 10 5 0
- \( d(12,6) = 100 \) 67 71 63 58 48 33 21 0

Monotonically decreasing distance functions: \( d(x,k) \) where
- \( d(1,6) = 46 \)
- \( d(3,6) = 0 \)

Distance functions which increase to a peak at a point
- \( k \) units from 1, then decrease: \( d(x,k) \) where
  - \( k \) 0 10 14
  - \( d(k,1) = 42 \) 22 77 27
  - \( d(k,1) = 73 \) 15 55 35
  - \( \Delta = 90.5 \) 23 23

\[ \text{Center of a Tree} \]

A center of a tree lies at the midpoint of the longest elementary chain in the tree.

**Finding Center of a Tree**

0. Choose arbitrarily a point \( X \) of the tree.
1. Find the vertex \( \text{farthest} \) from \( X \). Call this vertex \( V_1 \). (This will have degree 1.)
2. Find the vertex \( \text{farthest} \) from \( V_1 \). Call this vertex \( V_2 \). (This will also have degree 1.)
3. Find the midpoint \( X^c \) of the unique elementary path from \( V_1 \) to \( V_2 \). \( X^c \) will be the absolute center of the tree, and the vertex nearest to \( X^c \) will be the vertex center.

**Example**

Find the absolute & vertex centers of the tree:

Arbitrarily choose vertex 7. Label each vertex with its distance from vertex 7, to find that farthest from 7: (vertex 11)

Now label the vertices with their distances from vertex 11, to find that farthest from 11: vertex 3.
The midpoint of the chain from vertex 11 to vertex 3 is a distance 11 from vertex 11, on the edge [5,9].

The vertex center of the tree is at vertex #5, the vertex nearest to the absolute center.