Branch-&-Bound

algorithms for discrete optimization problems:

P: *find* $Z^* = \text{Minimum}\left\{cx : x \in S\right\}$

where the feasible set S is discrete,

for example, $S = \left\{x : Ax \geq b, x \in \{0,1\}\right\}$

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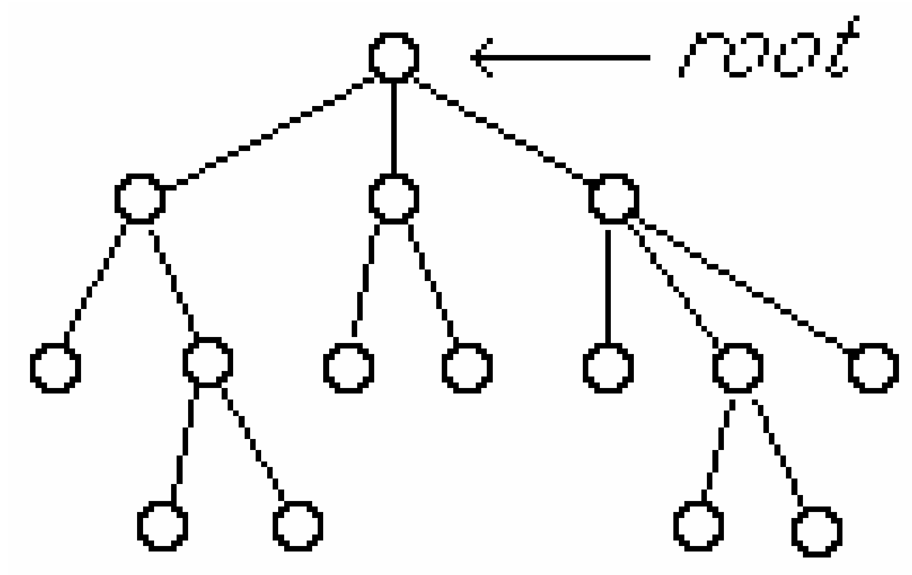
In order to visualize the branch-and-bound approach, we will use a ***search tree***:



- Each ***node*** of the search tree for a problem represents a *subset* of feasible solutions of the problem.
- The ***root*** of the tree represents the set of all feasible solutions of the problem.
- The ***descendents*** of each node of the tree represent a ***partition*** of the set represented by that node.

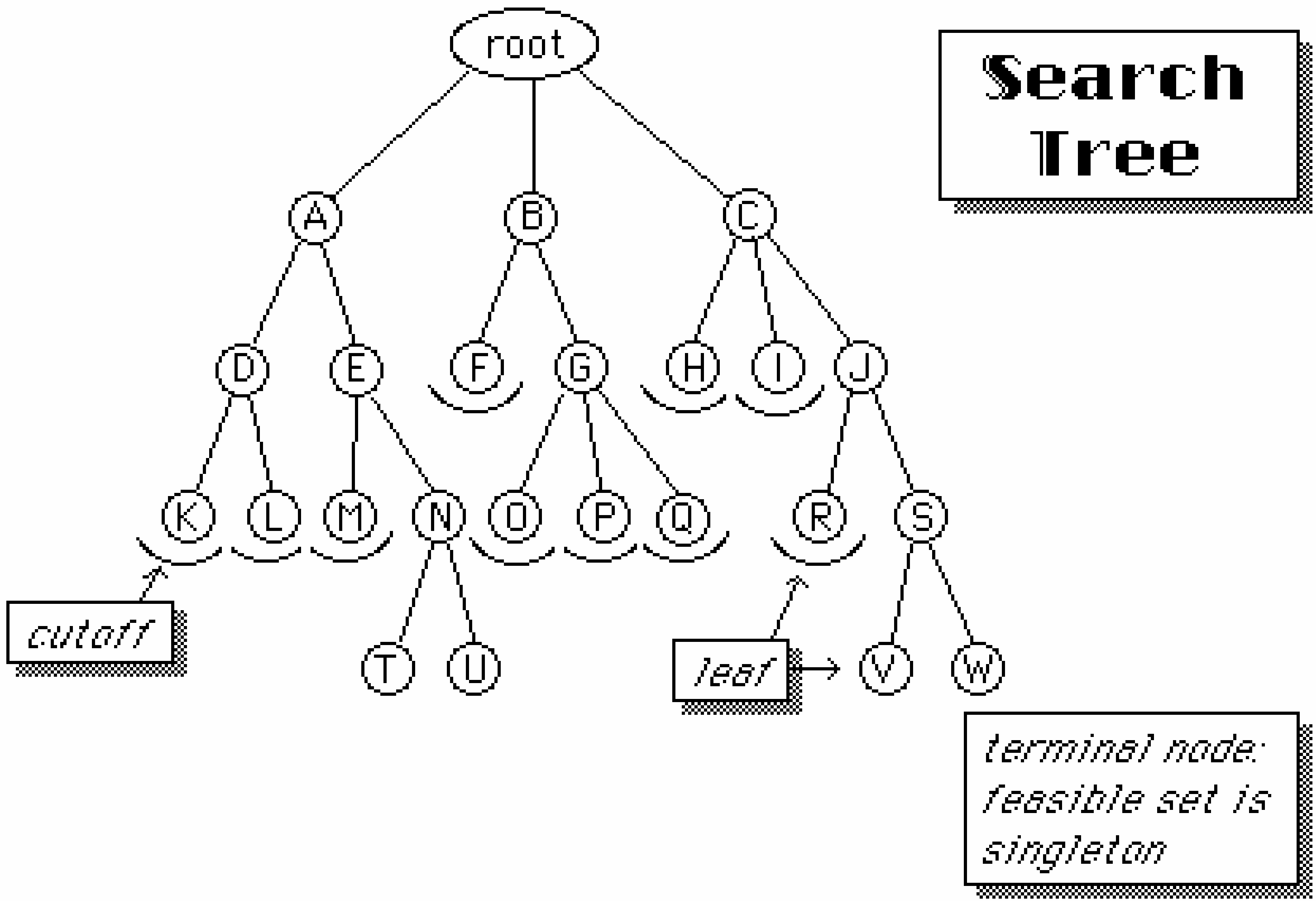
Creating descendents of a node with feasible set S_t is done by *partitioning*.

A collection of subsets $S_i, i = 1, 2, \ldots t$ of set S is a ***partition*** of S if

$$S_1 \cup S_2 \cup \cdots \cup S_t = S$$

and

$$S_i \cap S_j = \varnothing$$, i.e., they are *mutually disjoint*


Search Tree
root
A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
Q
R
S
T
U
V
W
cutoff
leaf
terminal node: feasible set is singleton

A ***leaf node*** is a node without descendents.

A ***terminal node*** is a leaf node whose feasible set consists of a single solution, i.e., a singleton.

A node can be ***fathomed*** (***pruned***) if one of three conditions are met:

- *Pruning by optimality*: $z_t = \min\{cx : x \in S_t\}$ has been computed.
- *Pruning by bound*: the inequality $\underline{z} \leq \min\{cx : x \in S_t\}$ has been proved.
- *Pruning by infeasibility*: $S_t = \varnothing$

in which case it is a leaf node, i.e., it is not necessary to create descendents (*branch*).

If no pruning is done, then all the terminal nodes must be determined by *complete enumeration*.

By pruning, we perform *implicit enumeration*, i.e., we avoid explicitly finding each terminal node.

Pruning by bound, i.e., proving that $\underline{z} \leq \min\{cx : x \in S_t\}$, is usually achieved by using a ***relaxation*** of $\min\{cx : x \in S_t\}$.

Consider a constrained optimization problem

$$\mathbf{P:} \quad z^* = \min\{cx \mid x \in S\}$$

and a problem

$$\mathbf{P^R:} \quad z^R = \min\{f(x) \mid x \in T\}$$

The problem **P**^{R} is a **relaxation** of problem **P** if:

- $S \subseteq T$, i.e., every x feasible in P is also feasible in P^R,

and

- $f(x) \le cx \quad \forall x \in S$

Proposition: If **P**^{R} is a relaxation of **P**, then its optimal value is a *lower* bound of the optimal value of **P**:

$$z^R \leq z^*.$$

Therefore, if at some node we solve a relaxation P^R and find z^R and can show that

$z^R \geqslant z^*$, we can ***prune*** the node.

To be useful, $\mathbf{P^R}$ should be easier to solve than **P**.

Linear Programming Relaxation of Integer (& Mixed-Integer) LP

The most common relaxation of IP & MIP problems is the **LP relaxation**, in which integer restrictions are removed.
Suppose that

$$P: \quad z = \min\left\{ cx \mid Ax \geq b, x \in Z_+^n \right\}$$

where Z_+^n is the set of n-dimensional vectors of non-negative *integers*.
The ***LP relaxation*** is

$$P^{LP}: \quad z^{LP} = \min\left\{ cx \mid Ax \geq b, x \in R_+^n \right\}$$

where R_+^n is the set of n-dimensional real non-negative vectors.

Note: in the above definition of ***relaxation***, let

$$f(x) = cx \text{ and}$$

$$S \equiv \left\{ x \mid Ax \geq b, x \in Z_+^n \right\} \quad \& \quad T \equiv \left\{ x \mid Ax \geq b, x \in R_+^n \right\}$$

so that $S \subset T$

That is, while the objective functions of P & P^{LP} are the same, relaxing the integer restrictions of an IP adds feasible solutions to the problem, so that a lower minimum might be found.