Bernoulli & Poisson Processes: Exercises

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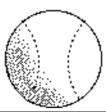
Example Two evenly-matched baseball teams are playing in the World Series, so that, for any one game, each team has **50%** probability of winning. The series is over as soon as one team wins ***four*** games.



What is the probability that the series is over in exactly four games?

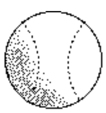
Solution

Solution

The probability that the National League team wins 4 straight games is $(0.5)^4 = 0.0625$
Likewise, the probability that the American League team wins 4 straight games is 0.0625

Thus, the probability that either the NL or AL team wins 4 straight games is $2 \times 0.0625 = 0.125$

return



The maximum number of games required in the series is seven. What is the probability that all seven games are required?

Solution

Solution

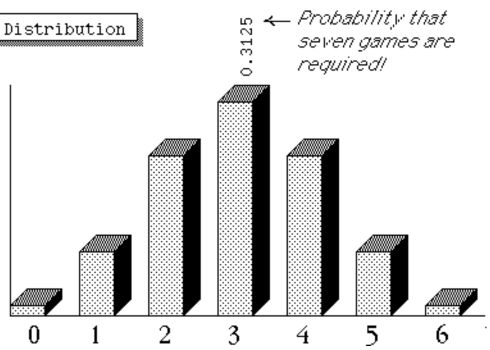
A seventh game is required if, after playing 6 games, the National League team (& therefore the AL team as well) has won 3 games.
The number of wins by the NL team when six games have been played will have the binomial distribution with n=6, p=0.5

return

Binomial Distribution

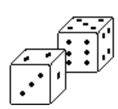
n= 6, p= 0.5

x	P[x]
0	0.015625
1	0.093750
2	0.234375
3	0.312500
4	0.234375
5	0.093750
6	0.015625



← *Probability that seven games are required!*

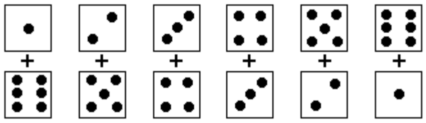
Example *What is the probability of throwing a 7 at least **twice** in **five** throws of a standard pair of dice?*



Solution

Solution There are 6x6=36 different outcomes when throwing the dice.

Six of these give a count of 7:



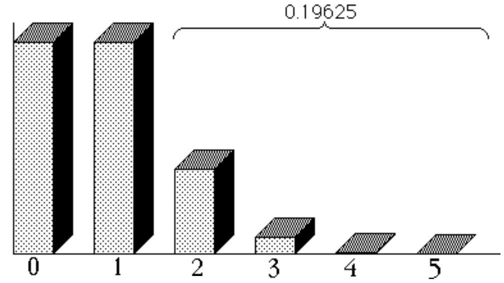
The number of 7's rolled in 5 throws of the dice will have ***binomial*** distribution, with n=5, p= $\frac{6}{36} = \frac{1}{6}$

return

Binomial Distribution

n= 5, p= 0.166667

x	P[x]
0	0.401877
1	0.401877
2	0.160751
3	0.032150
4	0.003215
5	0.000128



Example Opinion polls can often give erroneous results when a small sample is taken. Suppose that **60%** of the population of a town are opposed to a certain action, and that the others are for it. If **five** people are asked for their opinion, what is the probability that this small poll will show the majority **favor** the action (contrary to fact)?

Solution

Solution

Let X = # of opinions opposed to the action.
Then X has ***binomial*** distribution with n=5, p=0.6

What is P{X<2.5}, i.e.,
P{X is less than half of opinions polled}?

$$\sum_{x=0}^{2} \binom{5}{x} (0.6)^x (0.4)^{5-x}$$

return

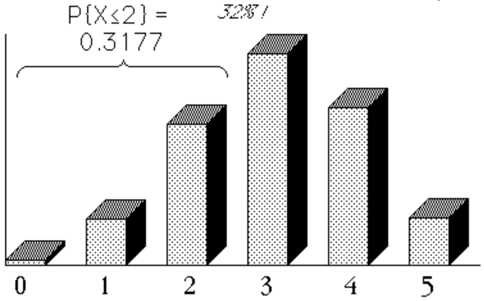
Binomial Distribution

n= 5, p= 0.6

That is, the probability of error is approximately 32%!

$\binom{5}{x} (.6)^x (.4)^{5-x}$

x	P[x]
0	0.01024
1	0.07680
2	0.23040
3	0.34560
4	0.25920
5	0.07776



Example A swimming pool builder guarantees completion of work within **30** working days. When properly scheduled, the work requires only **17** working days. He maintains a 13 day margin to allow for delays due to rain. During the month of August, records show that there is a **10%** chance of rain each day. For every day of rain, work will be suspended for one ***additional*** day. Under these circumstances, find the probability that the builder will not be able to meet the deadline.

Solution

Solution

Example In a radio quiz show, the listeners are invited to call in answers. The first caller to give the correct answer wins the prize. It has been noted that generally only **15%** of the callers have correct answers.

- *In a typical quiz show, what is the probability that the* ***eighth*** *caller wins the prize?*
- *What is the probability that the prize goes to one of the first* ***five*** *callers?*
- *What is the expected number of callers needed to win the prize?*

Solution

Solution Let T_1 = caller # of first winner

Then T_1 has the ***geometric*** distribution, with parameter p=0.15 *(or, equivalently, the* ***Pascal*** *distribution with parameters k=1, p=.15*

$E(T) = \frac{1}{p} = \frac{1}{0.15} = 6.6667$ *Expected # of callers needed to find winner*

return

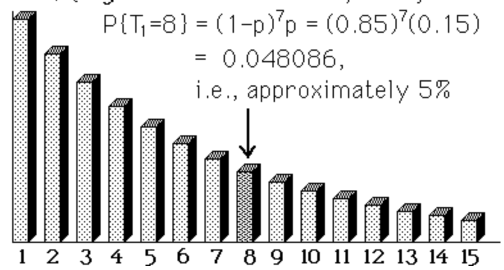
Pascal Distribution

k= 1, p= 0.15

Note: when k=1, this is the Geometric Distribution!

w	P[w]
1	0.150000
2	0.127500
3	0.108375
4	0.092118
5	0.078300
6	0.066555
7	0.056572
8	0.048086
9	0.040873
10	0.034742
11	0.029531
12	0.025101
13	0.021336
14	0.018135
15	0.015415

P{***eighth*** *caller wins the prize* } = $P\{T_1=8\} = (1-p)^7 p = (0.85)^7(0.15)$ = 0.048086, i.e., approximately 5%



Pascal Cumulative Distribution Function

k= 1, p= 0.15

Note: when k=1, this is the Geometric Distribution!

x	P{X≤x}
1	0.15000000
2	0.27750000
3	0.38587500
4	0.47799375
5	0.55629469
6	0.62285048
7	0.67942291
8	0.72750947
9	0.76838305
10	0.80312560
11	0.83265676
12	0.85775824
13	0.87909451
14	0.89723033
15	0.91264578

$P\{T \le 5\} = 0.55629$

Probability that winner is among first 5 callers

An advertisement for sales clerks is placed in a newspaper by a department store. Based on previous experience, the store expects applications to arrive randomly at an average rate of 2/day, for as long as the ad runs.

How long should the ad run, if the store wants to guarantee with ***98%*** *certainty that it will receive at least* ***six*** *applications?*

Solution

Solution

Assume that the applications arrive according to a **Poisson** process, with arrival rate $\lambda = 2$/day.

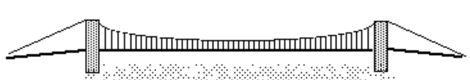
What is the smallest number of days ***t*** *such that*

$$P\{N_t \le 5\} = \sum_{x=0}^{5} \frac{(\lambda t)^x}{x!} e^{-\lambda t} < .02 \quad ?$$

return

t	$P\{N_t \le 5\}$
3	0.44567964
4	0.19123606
5	0.06708596
6	0.02034103
7	0.00553205
⋮	⋮

If the ad is run for only 3 days, then 6 applications are expected, but there is about 45% probability of fewer than 6! ***Seven*** days are required in order that the probability of receiving ***at least*** 6 applications is at least 98% *(in which case it will be nearly 99.5%)*



The number of cars passing over a toll bridge during the time interval 10 a.m. to 11 a.m. is **240**. The cars pass individually at random.

What is the probability that not more than 4 cars will pass during the 1-minute interval from 10:45 to 10:46 a.m.?

Solution

Solution Assume that the arrivals form a Poisson process, with arrival rate λ = 240/hr. = 4/minute.

$$P\{N_1 \le 4\} = \sum_{x=0}^{4} \frac{(\lambda t)^x}{x!} e^{-\lambda t} = (0.018315638) \sum_{x=0}^{4} \frac{4^x}{x!}$$

(0.018315638 = e^{-4})

= 0.62883694

return

Seaton's neighborhood grocery store experiences a demand for "Heat-N-Eat Steak" frozen dinners at the rate of 8 per 14-hour day. The store owner wants to be 99% certain that he does not run out of these dinners.

What is the least number of dinners that he should carry to maintain this level, assuming that the demand forms a Poisson process?

Solution

Solution

While it may be questionable whether the demand process is in fact ***Poisson***, let's make that assumption.

$$P\{N_{14} \le y\} = \sum_{x=0}^{y} \frac{(\lambda t)^x}{x!} e^{-\lambda t} = (3.354627 \times 10^{-4}) \sum_{x=0}^{y} \frac{8^x}{x!}$$

return

Poisson Cumulative Distribution Function

Mean: 8

x	P{X≤x}
⋮	⋮
5	0.19123606
6	0.31337428
7	0.45296081
8	0.59254734
9	0.71662426
10	0.81588579
11	0.88807600
12	0.93620280
13	0.96581930
14	0.98274301
15	0.99176899
16	0.99628198
⋮	⋮

To be 99% certain of avoiding a shortage, the store should stock at least 15 dinners!

A sampling inspection plan calls for the inspection of 50 items out of a lot of 1000 items. If there are no defective items in the sample, the lot is accepted. Otherwise it is rejected.

If a lot of 1% defective items is received, what is the probability that it will be accepted?

Solution

Solution

Let N_n = # of defective items found in the sample of size n

Then N_{50} has the ***Binomial*** distribution with n = 50, and p = 0.01

$$P\{N_n = x\} = \binom{n}{x} p^x (1-p)^{n-x}$$

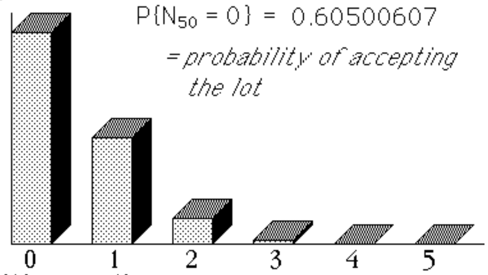
$$\Longrightarrow P\{N_{50} = 0\} = (0.01)^0 (0.99)^{50}$$

return

Binomial Distribution

n= 50, p= 0.01

x	P[x]
0	0.60500607
1	0.30555862
2	0.07561804
3	0.01222110
4	0.00145048
5	0.00013479



$P\{N_{50} = 0\}$ = 0.60500607

= probability of accepting the lot

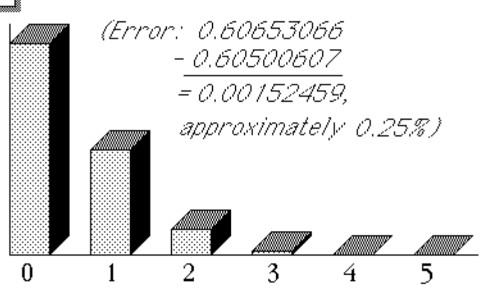
Sum of probabilities over the above range is 0.999989

Poisson Dist'n

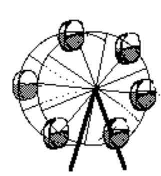
$P\{N_{50} = 0\}$ = 0.60653066

(Error: 0.60653066 − 0.60500607 = 0.00152459, approximately 0.25%)

x	P[x]
0	0.60653066
1	0.30326533
2	0.07581633
3	0.01263606
4	0.00157951
5	0.00015795



Sum of probabilities over the above range is 0.999986



A carnival ride, having six seats, lasts for five minutes. Persons wishing to ride arrive according to a Poisson process, with arrival rate 2/minute.

Suppose that a ride cycle has just begun, and there are no persons waiting. What is the probability that the next ride will be filled to capacity?

Solution

Solution Assume that the arrivals of persons who wish to ride form a Poisson process, with arrival rate $\lambda = 2$/minute.
Then N_t has the ***Poisson*** distribution:

$$P\{N_t = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

$$E(N_t) = \lambda t$$

The probability that, during the next five minutes, 6 or more persons arrive is $P\{N_5 \geq 6\} = 1 - P\{N_5 \leq 5\}$

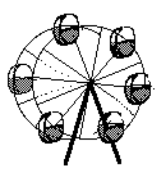
return

Poisson Cumulative Distribution Function

Mean: 10 ← λt (λ=2/min., t=5 min.)

x	P{X≤x}	P{X>x}
0	0.00004540	0.99995460
1	0.00049940	0.99950060
2	0.00276940	0.99723060
3	0.01033605	0.98966395
4	0.02925269	0.97074731
5	0.06708596	0.93291404
6	0.13014142	0.86985858
7	0.22022065	0.77977935
8	0.33281968	0.66718032
9	0.45792971	0.54207029
10	0.58303975	0.41696025

The probability that the next ride is filled to capacity is over 93%



Suppose that there are **4** customers already waiting, the last of which arrived **20** seconds ago. You hope to ride on the next cycle, which begins in **2** minutes, but you must first put your coat in a nearby locker, which will take **1** minute.

What is the probability that you will be able to ride in the next cycle?

⇦ ⇨ Solution

Solution You will be able to ride in the next cycle if, during the next minute while you are gone, no more than 1 person arrives for the ride.

$$P\{N_1 \leq 1\} = \sum_{x=0}^{1} \frac{(2)^x}{x!} e^{-2} = e^{-2} \sum_{x=0}^{1} \frac{(2)^x}{x!}$$

$$= 0.1353353 \left(\frac{2^0}{0!} + \frac{2^1}{1!}\right)$$

$$= 0.1353353\,(1 + 2)$$

$$= 0.40600585$$

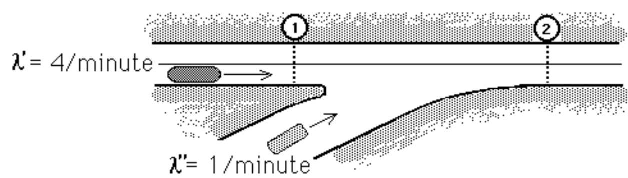
return

Poisson Cumulative Distribution Function

Mean: 2 ← λt (λ=2/min., t=1 min.)

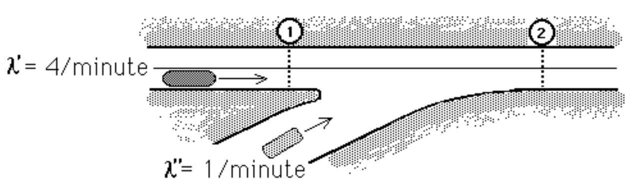
x	P{X≤x}	P{X>x}
0	0.13533528	0.86466472
1	0.40600585	0.59399415
2	0.67667642	0.32332358
3	0.85712346	0.14287654

The probability that 1 or fewer other persons arrive during the next minute is about 41%



Eastbound traffic on I-80 and the Dubuque Street entrance ramp form Poisson processes, with rates as shown. Counters #1 & #2 have been placed to measure the traffic before & after the ramp, respectively.

⇦ ⇨



What is the probability that

exactly 20 vehicles cross counter #1 during a 5-minute interval?

⇨ Solution

Solution

The number of vehicles passing counter #1 during a 5-minute interval with have the ***Poisson*** distribution, with mean value $\lambda t = 20$, where $\lambda = 4$/minute and t=5 minutes.

$$P\{N_t = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

$$P\{N_5 = 20\} = \frac{20^{20}}{20!} e^{-20} = \frac{1.048576\times10^{26}}{2.4329\times10^{18}} \times 2.06115\times10^{-9}$$

$$= 0.088835$$

return

Poisson Dist'n

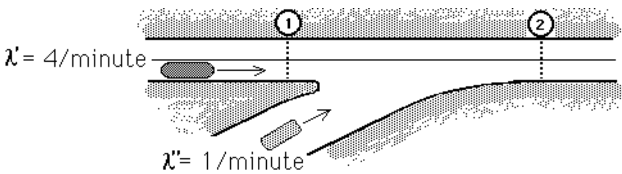
Mean: 20 ← λt (λ = 4/min., t=5 min.)

x	P[x]
15	0.05164885
16	0.06456107
17	0.07595420
18	0.08439355
19	0.08883532
20	0.08883532
21	0.08460506
22	0.07691369
23	0.06688147
24	0.05573456
25	0.04458765

The probability that ***exactly*** *20 vehicles pass counter #1 is less than 9%*

Poisson Cumulative Distribution Function   Mean: 20

x	P{X≤x}	P{X>x}
10	0.01081172	0.98918828
11	0.02138682	0.97861318
12	0.03901199	0.96098801
13	0.06612764	0.93387236
14	0.10486428	0.89513572
15	0.15651313	0.84348687
16	0.22107420	0.77892580
17	0.29702840	0.70297160
18	0.38142195	0.61857805
19	0.47025727	0.52974273
20	0.55909258	0.44090742
21	0.64369765	0.35630235
22	0.72061134	0.27938866
23	0.78749282	0.21250718
24	0.84322738	0.15677262
25	0.88781503	0.11218497



What is the probability that

the next vehicle to pass counter #2 arrives during the next 15 seconds?

Solution

Solution

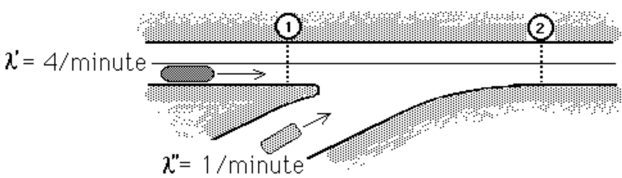
T_1 = time until next car passes counter #2 has an ***exponential*** distribution with parameter $\lambda = (\lambda' + \lambda'') =$ 5/minute.

CDF $P\{T_1 \le t\} = F(t) = 1 - e^{-\lambda t}$

$F(0.25) = P\{T_1 \le 0.25\} = 1 - e^{-5(0.25)}$
$= 0.713495203$

$\frac{1}{4}$ minute = 15 sec.

return



What is the probability that

The time until 10 vehicles pass counter #2 is less than 2 minutes?

Solution

Solution

T_{10} = time of arrival of 10^{th} vehicle has the ***Erlang*** (or Gamma) distribution.

$P\{T_{10} < 2\} = ?$

This can be found by computing

$P\{N_2 \ge 10\}$

where N_2 = # arrivals at time 2 has the ***Poisson*** distribution

return

$P\{N_2 \ge 10\} = 1 - P\{N_2 \le 9\}$

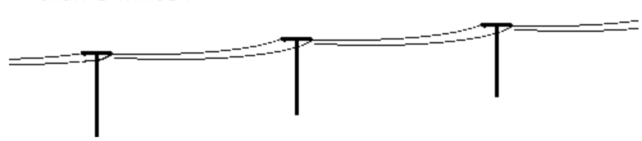
Poisson Cumulative Distribution Function

Mean: 10

x	P{X≤x}	P{X>x}
0	0.00004540	0.99995460
1	0.00049940	0.99950060
2	0.00276940	0.99723060
3	0.01033605	0.98966395
4	0.02925269	0.97074731
5	0.06708596	0.93291404
6	0.13014142	0.86985858
7	0.22022065	0.77977935
8	0.33281968	0.66718032
9	0.45792971	0.54207029
10	0.58303975	0.41696025

Example In a transmission line, there is a fault in the insulation every 2.5 miles, on the average.

What is the probability of having 2 faults in less than 6 miles?



Solution

Solution

The random variable

T_k = location of the k^{th} fault

has the ***k-Erlang*** distribution with parameter.

$\lambda = {}^1/_{2.5}$ miles = 0.4/mile

$P\{T_2 \le 6\} = P\{N_6 \ge 2\} = 1 - P\{N_6 \le 1\}$

where

N_x = # faults in length x miles

has the ***Poisson*** distribution.

return

Poisson Cumulative Distribution Function

Mean: 2.5

x	P{X≤x}	P{X>x}
0	0.08208500	0.91791500
1	0.28729750	0.71270250
2	0.54381312	0.45618688
3	0.75757613	0.24242387
4	0.89117802	0.10882198
5	0.95797896	0.04202104
6	0.98581269	0.01418731
⋮	⋮	⋮

$P\{N_6 \ge 2\} = 1 - P\{N_6 \le 1\}$