Suppose that we wish to select the location of a single facility, anywhere in the plane, to serve a set of demand points.

Given, for each of demand points $j = 1, 2, \ldots, n$:

- $(x_j, y_j)$ coordinates of the point
- $\beta_j$ cost per unit volume per unit distance
- $w_j$ volume of shipments per unit time

Find coordinates of the source facility, $(x, y)$, which will minimize the total shipping cost per unit time:

$$\text{Minimize } C(x, y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}$$

assuming "straight-line", Euclidean distances!

Weber's Problem
Tie together in a knot ("X") n strings of equal length L

Attach a bucket with weight \( W_i \) at the end of string i.
Theorem

The function:

\[ C(x,y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2} \]

is convex in \((x,y)\)

A necessary condition for \((X^*,Y^*)\) to minimize

\[ C(x,y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2} \]

is

\[
\begin{align*}
\frac{\partial}{\partial x} C(X^*,Y^*) &= 0 \\
\frac{\partial}{\partial y} C(X^*,Y^*) &= 0
\end{align*}
\]

That is, \((X^*,Y^*)\) should be a "stationary point" of the function \(C\).
This condition yields the equations

\[
\begin{align*}
\sum_{j=1}^{n} \frac{\beta_j w_j (X^* - x_j)}{\sqrt{(X^* - x_j)^2 + (Y^* - y_j)^2}} &= 0 \\
\sum_{j=1}^{n} \frac{\beta_j w_j (Y^* - y_j)}{\sqrt{(X^* - x_j)^2 + (Y^* - y_j)^2}} &= 0
\end{align*}
\]

which, unfortunately, we cannot solve analytically for the values of $X^*$ and $Y^*$!

For convenience, define a distance function for each $j$:

\[d_j(X, Y) = \sqrt{(X - x_j)^2 + (Y - y_j)^2}\]

Necessary conditions for optimality

\[
\begin{align*}
\sum_{j=1}^{n} \frac{\beta_j w_j (X^* - x_j)}{d_j(X^*, Y^*)} &= 0 \\
\sum_{j=1}^{n} \frac{\beta_j w_j (Y^* - y_j)}{d_j(X^*, Y^*)} &= 0
\end{align*}
\]
Rearrange terms:

\[
\begin{align*}
X^* \sum_{j=1}^{n} \frac{\beta_j w_j}{d_j(X^*,Y^*)} &= \sum_{j=1}^{n} \frac{\beta_j w_j X_j}{d_j(X^*,Y^*)} \\
Y^* \sum_{j=1}^{n} \frac{\beta_j w_j}{d_j(X^*,Y^*)} &= \sum_{j=1}^{n} \frac{\beta_j w_j Y_j}{d_j(X^*,Y^*)}
\end{align*}
\]

Necessary Conditions for the Optimality of \((X^*,Y^*)\)

\[
\begin{align*}
X^* &= \frac{\sum_{j=1}^{n} \frac{\beta_j w_j X_j}{d_j(X^*,Y^*)}}{\sum_{j=1}^{n} \frac{\beta_j w_j}{d_j(X^*,Y^*)}} \\
Y^* &= \frac{\sum_{j=1}^{n} \frac{\beta_j w_j Y_j}{d_j(X^*,Y^*)}}{\sum_{j=1}^{n} \frac{\beta_j w_j}{d_j(X^*,Y^*)}}
\end{align*}
\]

Note: \(X^*\) and \(Y^*\) actually appear on both sides of the equations!
We will use a "successive substitution" method using these equations to find \( X^* \) & \( Y^* \)

\[
X^* = \frac{\sum_{j=1}^{n} \frac{\beta_j w_j x_j}{d_j(X^*, Y^*)}}{\sum_{j=1}^{n} \frac{\beta_j w_j}{d_j(X^*, Y^*)}}
\]
\[
Y^* = \frac{\sum_{j=1}^{n} \frac{\beta_j w_j y_j}{d_j(X^*, Y^*)}}{\sum_{j=1}^{n} \frac{\beta_j w_j}{d_j(X^*, Y^*)}}
\]

Suppose, at iteration \( k \), we have an approximate solution \((X^k, Y^k)\).
We obtain an improved approximate solution \((X^{k+1}, Y^{k+1})\) by

\[
X^{k+1} = \frac{\sum_{j=1}^{n} \frac{\beta_j w_j x_j}{d_j(X^k, Y^k)}}{\sum_{j=1}^{n} \frac{\beta_j w_j}{d_j(X^k, Y^k)}} \quad \& \quad Y^{k+1} = \frac{\sum_{j=1}^{n} \frac{\beta_j w_j y_j}{d_j(X^k, Y^k)}}{\sum_{j=1}^{n} \frac{\beta_j w_j}{d_j(X^k, Y^k)}}
\]
Weiszfeld Algorithm

Starting with an initial "guess" \((X^0, Y^0)\), we will generate a sequence of approximate solutions, \((X^1, Y^1), (X^2, Y^2), (X^3, Y^3), \ldots\) which converge to the optimal facility location \((X^*, Y^*)\).

We terminate the method when two successive approximate solutions are "close enough", i.e.,

\[
|X^{k+1} - X^k| + |Y^{k+1} - Y^k| < \epsilon \approx 0
\]

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Example

<table>
<thead>
<tr>
<th>Customer</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location</td>
<td>(0,3)</td>
<td>(2,4)</td>
<td>(4,3)</td>
<td>(1,0)</td>
</tr>
<tr>
<td>Rqmt. (Ton/wk)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Cost/ton-mile is same for all customers

Where should a supply facility be located so that total shipping cost per week is minimized?

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A good starting point is the centroid, i.e., the weighted average of the customer coordinates.

\[ x^0 = \frac{4}{\sum_{j=1}^{4} \beta_j w_j} \sum_{j=1}^{4} \beta_j w_j x_j \quad \text{and} \quad y^0 = \frac{4}{\sum_{j=1}^{4} \beta_j w_j} \sum_{j=1}^{4} \beta_j w_j y_j \]

\[ \beta_j = 1 \quad \forall j \]

<table>
<thead>
<tr>
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<th>3</th>
<th>4</th>
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<td>(4,3)</td>
<td>(1,0)</td>
</tr>
<tr>
<td>Rqmt.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ x^0 = \frac{4}{\sum_{j=1}^{4} \beta_j w_j} \sum_{j=1}^{4} \beta_j w_j x_j = \frac{1 \times 0 + 2 \times 2 + 3 \times 4 + 2 \times 1}{1 + 2 + 3 + 2} = 2.25 \]

\[ y^0 = \frac{4}{\sum_{j=1}^{4} \beta_j w_j} \sum_{j=1}^{4} \beta_j w_j y_j = \frac{1 \times 3 + 2 \times 4 + 3 \times 3 + 2 \times 1}{1 + 2 + 3 + 2} = 2.5 \]
Now compute distance from \((X^0, Y^0)\) to each customer:

\[
\begin{align*}
    d_1 &= \sqrt{\left(\frac{9}{4} - 0\right)^2 + \left(\frac{5}{2} - 3\right)^2} \approx 2.305 \\
    d_2 &= \sqrt{\left(\frac{9}{4} - 2\right)^2 + \left(\frac{5}{2} - 4\right)^2} \approx 1.521 \\
    d_3 &= \sqrt{\left(\frac{9}{4} - 4\right)^2 + \left(\frac{5}{2} - 3\right)^2} \approx 1.820 \\
    d_4 &= \sqrt{\left(\frac{9}{4} - 1\right)^2 + \left(\frac{5}{2} - 0\right)^2} \approx 2.795
\end{align*}
\]

\[\text{Shipping Cost:}\]

\[
1(2.305) + 2(1.521) + 3(1.820) + 2(2.795) = 16.397
\]
Apply our successive substitution method to (we hope!) obtain a better approximate solution:

\[ X^1 = \frac{\sum \frac{w_j x_j}{d_j}}{\sum \frac{w_j}{d_j}} = \frac{\frac{1\times0}{d_1} + \frac{2\times2}{d_2} + \frac{3\times4}{d_3} + \frac{2\times1}{d_4}}{4.113} \approx \frac{10.373}{4.113} = 2.522 \]

\[ Y^1 = \frac{\sum \frac{w_j y_j}{d_j}}{\sum \frac{w_j}{d_j}} = \frac{\frac{1\times3}{d_1} + \frac{2\times4}{d_2} + \frac{3\times3}{d_3} + \frac{2\times0}{d_4}}{4.113} \approx \frac{11.506}{4.113} = 2.798 \]

\[ d_j = \sqrt{[2.522-x_j]^2 + [2.798-y_j]^2} \]

\[ d_1 = 2.530 \]
\[ d_2 = 1.310 \]
\[ d_3 = 1.492 \]
\[ d_4 = 3.185 \]

Shipping cost:

\[ 1(2.53) + 2(1.31) + 3(1.492) + 2(3.185) = 15.996 \]

reduction of 2.4%
Distance between initial and improved solution:

0.421

Perform additional iterations, until distance moved is "sufficiently small"


**Iteration # 2**

Facility location at X = 2.41669, Y = 2.79785

Distances to demand pts:

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>D(i)</td>
<td>2.42503</td>
<td>1.27229</td>
<td>1.59626</td>
<td>3.13603</td>
</tr>
<tr>
<td>W(i)×D(i)</td>
<td>2.42503</td>
<td>2.54457</td>
<td>4.79879</td>
<td>6.27206</td>
</tr>
</tbody>
</table>

Total cost is 15.0305

New location is at X = 2.51012, Y = 2.92419

Rectilinear distance moved is 0.21987

---

**Iteration # 3**

Facility location at X = 2.51012, Y = 2.92419

Distances to demand pts:

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>D(i)</td>
<td>2.51126</td>
<td>1.19063</td>
<td>1.49181</td>
<td>3.2911</td>
</tr>
<tr>
<td>W(i)×D(i)</td>
<td>2.51126</td>
<td>2.38126</td>
<td>4.47542</td>
<td>6.5822</td>
</tr>
</tbody>
</table>

Total cost is 15.9501

New location is at X = 2.55738, Y = 2.96049

Rectilinear distance moved is 0.0925647

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Facility location at $X = 2.55738$, $Y = 2.96949$

Distances to demand pts:

$$
\begin{array}{cccccc}
  & 1 & 2 & 3 & 4 \\
D[i] & 2.55738 & 1.1716 & 1.44294 & 3.3631 \\
WT[i]\times D[i] & 2.55738 & 2.34319 & 4.32361 & 6.75622 \\
\end{array}
$$

Total cost is $15.9359$

New location is at $X = 2.58231$, $Y = 2.98276$

Rectilinear distance moved is $0.0381954$

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---

Facility location at $X = 2.558231$, $Y = 2.98276$

Distances to demand pts:

$$
\begin{array}{cccccc}
  & 1 & 2 & 3 & 4 \\
D[i] & 2.558237 & 1.17212 & 1.41738 & 3.37647 \\
WT[i]\times D[i] & 2.558237 & 2.34424 & 4.25539 & 6.75294 \\
\end{array}
$$

Total cost is $15.9329$

New location is at $X = 2.59867$, $Y = 2.98533$

Rectilinear distance moved is $0.0188786$

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Facility location at $X = 2.59667, Y = 2.98528$

Distances to demand pts:

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(i)$</td>
<td>2.59671</td>
<td>1.17715</td>
<td>1.48341</td>
<td>3.38544</td>
</tr>
<tr>
<td>$W(i) \times D(i)$</td>
<td>2.59671</td>
<td>2.35429</td>
<td>4.21023</td>
<td>6.77069</td>
</tr>
</tbody>
</table>

Total cost is 15.9321

New location is at $X = 2.60557, Y = 2.98478$

Rectilinear distance moved is $0.0094046 < 0.01$ (stopping criterion)

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The Optimal Location for the Supply Facility
Path Followed by the Successive Substitution Method

final solution

x = 2.60557, y = 2.98478

initial guess (centroid)

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