Function of One Variable

Suppose that \( f(x) \), \( f'(x) \), and \( f''(x) \) exist on the closed interval \([a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}\). If \( x^* \) and \( x \) are any two distinct points in \([a,b]\), then there exists a point \( z \) between \( x^* \) and \( x \) such that

\[
f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{f''(z)}{2}(x - x^*)^2
\]
Taylor's Formula

\[
f(x) = f(x^*) + f'(x^*) (x - x^*) + \frac{f''(z)}{2} (x - x^*)^2
\]

If \( f''(x) > 0 \) for all \( x \), and \( f'(x^*) = 0 \), then Taylor's Formula tells us that

\[ f(x) = f(x^*) + 0 + \text{a positive number} \quad > \quad f(x^*) \]

That is, \( x^* \) is the point that minimizes the function \( f \).

© D.L. Bricker, U. of IA, 1999

**Critical Point**

The point \( x^* \) is a *critical point* of a function \( f \) if \( f'(x^*) \) exists and equals zero.

*(stationary point)*

© D.L. Bricker, U. of IA, 1999
**Function of Several Variables**

**Gradient**

vector of first partial derivatives

\[ \nabla f(x) = \begin{bmatrix} \frac{df(x)}{dx_1}, \frac{df(x)}{dx_2}, \ldots, \frac{df(x)}{dx_n} \end{bmatrix} \]

**Hessian**

matrix of second partial derivatives

\[ \nabla^2 f(x) = \begin{bmatrix} \frac{d^2f(x)}{dx_1^2} & \frac{d^2f(x)}{dx_1dx_2} & \ldots & \frac{d^2f(x)}{dx_1dx_n} \\ \frac{d^2f(x)}{dx_2dx_1} & \frac{d^2f(x)}{dx_2^2} & \ldots & \frac{d^2f(x)}{dx_2dx_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^2f(x)}{dx_ndx_1} & \frac{d^2f(x)}{dx_ndx_2} & \ldots & \frac{d^2f(x)}{dx_n^2} \end{bmatrix} \]

---

Suppose that \( x^* \) and \( x \) are points in \( \mathbb{R}^n \) and that \( f(x) \) is a function of \( n \) variables with continuous first and second partial derivatives on some open set containing the line segment \([x^*, x]\) joining \( x^* \) and \( x \). Then there exists a \( z \in [x^*, x] \) such that

\[
f(x) = f(x^*) + \nabla f(x^*) \cdot (x - x^*) + \frac{1}{2} (x - x^*) \cdot \nabla^2 f(z) (x - x^*)
\]
QUADRATIC FORM

\[ f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j = x^T A x \]

A is not unique, but we can choose A to be symmetric (\( A = \frac{1}{2} \nabla^2 f(x) \))

\[
x_1^2 + x_1 x_2 + 3x_2^2 = [x_1 \ x_2] \begin{bmatrix} 1 & 1/2 \\ 1/2 & 3 \end{bmatrix} [x_1 \ x_2]
\]

\[ A_{ij} = \text{coefficient of } x_i^2 \]
\[ A_{ij} = \frac{1}{2} \text{ of coefficient of } x_i x_j \]

\( \odot D.L. Bricker, U. of IA, 1999 \)

Which are quadratic forms?

\[ x_1 + 2x_2^2 \]
\[ x_1 x_2 \]
\[ 3x_1^2 - x_1 x_2 \]
\[ x_1 x_2 - x_2 x_3 + x_1 x_3 \]

\( \odot D.L. Bricker, U. of IA, 1999 \)
\[
\begin{align*}
x_1^2 + x_2^2 &= x_T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \\
&> 0 \text{ for } x \neq 0 \\
&\text{positive definite}
\end{align*}
\]
\[
\begin{align*}
x_1^2 + 2x_1x_2 + x_2^2 &= x_T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x \\
&= (x_1 + x_2)^2 \geq 0 \text{ for all } x \\
&\text{positive semidefinite}
\end{align*}
\]
\[
\begin{align*}
x_1^2 - x_2^2 &= x_T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x \\
&\text{indefinite}
\end{align*}
\]

**Positive Definite**

A square symmetric matrix \( A \) is positive definite if

\[ x^T A x > 0 \text{ for all } x \neq 0 \]

**Note:** a symmetric matrix whose entries are all positive need not be positive definite.

**Example:** \[ A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \]

Let \( x = [1, -1] \):

\[ \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix} = -6 < 0 \]
Positive Definite

A symmetric matrix with some negative elements may be positive definite.

Example: \( A = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} \)

\[ x^t A x = x_1^2 - 2x_1x_2 + 4x_2^2 = (x_1 - x_2)^2 + 3x_2^2 > 0 \]

for all \( x \neq 0 \)

© D. L. Bricker, U. of IA, 1999

Positive Semidefinite

A square symmetric matrix \( A \) is positive semidefinite if \( x^t A x \geq 0 \) for all \( x \)

© D. L. Bricker, U. of IA, 1999
A square symmetric matrix $A$ is negative definite if

$x^tAx < 0 \text{ for all } x \neq 0$

---

A square symmetric matrix $A$ is negative semidefinite if $x^tAx \leq 0 \text{ for all } x$
**Indefinite**  
A square symmetric matrix $A$ is indefinite if

\[ \exists x^+ \text{ such that } (x^+)^t A x^+ > 0, \]

and

\[ \exists x^- \text{ such that } (x^-)^t A x^- < 0 \]

i.e., if it is neither positive semidefinite nor negative semidefinite.

---

**Diagonal Matrices**

A diagonal matrix $D$ is

- **positive definite** if $D_i > 0$ for all $i$
- **positive semidefinite** if $D_i \geq 0$ for all $i$
- **negative definite** if $D_i < 0$ for all $i$
- **negative semidefinite** if $D_i \leq 0$ for all $i$

\[ x^t D x = \sum_{i=1}^{n} D_i x_i^2 \]
Suppose that a symmetric matrix $A$ is reduced to upper triangular form by use of the elementary row operation

- Add to any row a scalar multiple of another row without using
- Multiply any row of the matrix by a (positive or negative) scalar
- Interchange two rows of the matrix

Then $A$ is

- **positive definite** if $U_i > 0 \; \forall \; i$
- **positive semidefinite** if $U_i \geq 0 \; \forall \; i$
- **negative definite** if $U_i < 0 \; \forall \; i$
- **negative semidefinite** if $U_i \leq 0 \; \forall \; i$
WHY?

Consider the quadratic form $x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$

$x^T A x = x^T L D L^T x = [L^T x]^T D [L^T x] = y^T D y = \sum_{i=1}^{n} D_i y_i^2$

where $y = L^T x$

If $D_i \geq 0$, then, $x^T A x \geq 0$ for all $x$

If $D_i > 0$, $x^T A x > 0$ for all $x \neq 0 (\Rightarrow y \neq 0)$

etc.

$A$ is positive
semidefinite

$A$ is positive
definite