Examples

Water Allocation

Production Planning

Transportation Problem (random demand)

2-Stage Stochastic Programming
EXAMPLE

A water system manager must allocate water from a stream to three users:
- municipality
- industrial concern
- agricultural sector

<table>
<thead>
<tr>
<th>Use</th>
<th>Request</th>
<th>Net Benefit per unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Municipality</td>
<td>2</td>
<td>100</td>
</tr>
<tr>
<td>2. Industrial</td>
<td>3</td>
<td>50</td>
</tr>
<tr>
<td>3. Agricultural</td>
<td>5</td>
<td>30</td>
</tr>
</tbody>
</table>

Let $X_i =$ amount of water allocated to use #i

The optimal allocation might be found by solving the LP:

$$\begin{align*}
\text{Max} & \quad 100X_1 + 50X_2 + 30X_3 \\
\text{subject to} & \quad X_1 + X_2 + X_3 \leq Q \\
& \quad 0 \leq X_1 \leq 2 \\
& \quad 0 \leq X_2 \leq 3 \\
& \quad 0 \leq X_3 \leq 5
\end{align*}$$

But the decision must be made before the quantity $Q$ of the available water is known!
Max $100X_1 + 50X_2 + 30X_3$ subject to $X_1 + X_2 + X_3 \leq Q$

$0 \leq X_1 \leq 2$
$0 \leq X_2 \leq 3$
$0 \leq X_3 \leq 5$

Random variable with known probability distribution

How should the water be allocated before the quantity available is known?

<table>
<thead>
<tr>
<th>Streamflow Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Use</th>
<th>Request</th>
<th>Loss per unit shortfall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Municipality</td>
<td>2</td>
<td>250</td>
</tr>
<tr>
<td>2. Industrial</td>
<td>3</td>
<td>75</td>
</tr>
<tr>
<td>3. Agricultural</td>
<td>5</td>
<td>60</td>
</tr>
</tbody>
</table>

If more water is promised than can be later delivered, then a loss results from the need either to acquire alternative sources &/or to reduce consumption.

What is the "optimal" quantity to allocate to each use, if $Q$ is not yet known?
Example

Production Planning with Uncertain Resources

Par, Inc., a manufacturer of golf bags, must schedule production for the next quarter.

<table>
<thead>
<tr>
<th>Product</th>
<th>Cutting &amp; Dyeing</th>
<th>Sewing</th>
<th>Finishing</th>
<th>Inspect Package</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>$\frac{7}{10}$ hr</td>
<td>$\frac{1}{2}$ hr</td>
<td>1 hr</td>
<td>$\frac{1}{10}$ hr</td>
</tr>
<tr>
<td>Deluxe</td>
<td>1 hr</td>
<td>$\frac{5}{6}$ hr</td>
<td>$\frac{2}{3}$ hr</td>
<td>$\frac{1}{4}$ hr</td>
</tr>
</tbody>
</table>

The company can sell as many bags as can be produced at a profit of $10 per standard bag and $9 per deluxe bag.
Max \[ 10X_1 + 9X_2 \]
subject to
\[
\begin{align*}
7/10X_1 + X_2 & \leq 630 \\
1/2X_1 + 5/6X_2 & \leq 600 \\
X_1 + 2/3X_2 & \leq 708 \\
1/10X_1 + 1/4X_2 & \leq 135 \\
X_1 \geq 0, X_2 \geq 0
\end{align*}
\]

Based upon current commitments, the hours available in each department for the next quarter are computed. However, the firm has submitted bids on two contracts, which if successful would reduce the hours available for producing golf bags.

<table>
<thead>
<tr>
<th>Contract</th>
<th>probability</th>
<th>Production Hours Reqd</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>C&amp;D</td>
</tr>
<tr>
<td>#1</td>
<td>50%</td>
<td>50</td>
</tr>
<tr>
<td>#2</td>
<td>40%</td>
<td>30</td>
</tr>
</tbody>
</table>

A production schedule for standard & deluxe bags must be chosen before learning which contracts, if any, were awarded to the firm. Afterwards, the production schedule may be modified somewhat, but extra costs are incurred in doing so...
For each scenario, we compute the available hours in each department (subtracting the hours used to fill any contracts which are won)

<table>
<thead>
<tr>
<th>Dept.</th>
<th>scenario #0</th>
<th>scenario #1</th>
<th>scenario #2</th>
<th>scenario #3</th>
</tr>
</thead>
<tbody>
<tr>
<td>C&amp;D</td>
<td>630</td>
<td>580</td>
<td>600</td>
<td>550</td>
</tr>
<tr>
<td>SEW</td>
<td>600</td>
<td>560</td>
<td>550</td>
<td>510</td>
</tr>
<tr>
<td>FIN</td>
<td>708</td>
<td>628</td>
<td>638</td>
<td>558</td>
</tr>
<tr>
<td>I&amp;P</td>
<td>135</td>
<td>125</td>
<td>120</td>
<td>110</td>
</tr>
</tbody>
</table>

**Recourses**

- Scheduling overtime in C&D at $5/hr
  - SEW at $6/hr
  - FIN at $8/hr
  - I&P at $4/hr

  (only 100 hrs OT available in FIN)

- Schedule additional production of standard bags, at a reduced profit of $8/bag
**Linear Constraints**

\[ Ax + By = b \]
\[ x \geq 0, \ y \geq 0 \]

**Sequence of Events**

- \( x \) is selected by the decision-maker
- the random variable \( b \) is observed
- the decision-maker must choose \( y \) so as to satisfy constraint, i.e.
  \[ By = b - Ax \]
Costs Incurred

$cx + dy$

Second-Stage Problem

$\phi(x, b) = \text{Minimum } dy$

s.t. $By = b - Ax$

$y \geq 0$

both $x$ & $b$ are fixed

Since $b$ is a random variable, so also is $\phi(x, b)$ for fixed $x$.

First-Stage Problem

Minimize $cx + E_b[\phi(x,b)]$

subject to $\phi(x,b) < \infty$

i.e., 2nd-Stage Problem should be feasible for all possible values of $b$

$E_b[\phi(x,b)]$ is the expected cost of the second stage, for fixed $x$.

This is generally a nonlinear, but convex, function of $x$. 
Suppose that the right-hand-side vector \(b\) is "drawn" from a finite set of possible RHSs \(\{b^1, b^2, \ldots, b^k\}\) with probabilities \(p_1, p_2, \ldots, p_k\).

Define a second-stage (recourse) vector for each of the possible RHSs: \(y^1, y^2, \ldots, y^k\).

Then the recourses must be selected so that given the first-stage decision \(x\), this system of equations is satisfied:

\[
\begin{align*}
Ax + By^1 &= b^1 \\
Ax + By^2 &= b^2 \\
&\quad \vdots \\
Ax + By^k &= b^k
\end{align*}
\]

The expected value of the second-stage cost is

\[p_1 dy^1 + p_2 dy^2 + \ldots + p_k dy^k\]
Minimize $cx + p_1y^1 + p_2y^2 + \ldots + p_ky^k$

subject to

- $Ax + By^1 = b^1$
- $Ax + By^2 = b^2$
- $Ax + By^3 = b^3$
- $\vdots$
- $Ax + By^k = b^k$

$x \geq 0, y^1 \geq 0, \ldots, y^k \geq 0$

Notice the block-angular structure of the coefficient matrix....

**Question:** Could the Dantzig-Wolfe decomposition technique be used in order to decompose this problem into smaller subproblems?

First-stage cost

plus

expected

2nd-stage cost
Dual of the 2-stage stochastic LP problem:

Maximize \( b^1u^1 + b^2u^2 + \ldots + b^ku^k \)
subject to \( A^Tu^1 + A^Tu^2 + \ldots + A^Tu^k \leq c \)
\( B^Tu^1 \leq p_1d \)
\( B^Tu^2 \leq p_2d \)
\[ \ldots \]
\( B^Tu^k \leq p_kd \)

_all variables unrestricted in sign_

This problem has a structure for which Dantzig-Wolfe decomposition is appropriate!
Subproblem for Block # i

Maximize \((b^i - \omega A^T) u^i - \alpha_i\)
subject to \(B^T u^i \leq p_i d\)

where \(\omega\) is the simplex multiplier vector for the linking constraints, and \(\alpha_i\) is the simplex multiplier vector for convexity constraint # i.

These subproblems all have the same matrix of constraint coefficients, and the constraint right-hand-side vectors are all scalar multiples of the same vector \(d\).

Solution

Water Allocation Problem

Define second-stage (recourse) variables

\(Y_i = \) amount of shortfall in water delivered to user \(i\)

Maximize

\[
\text{Max } 100X_1 + 50X_2 + 30X_3
\]

\[
- E_Q \left\{ \min 250Y_1 + 75Y_2 + 60Y_3 \right\}
\]

s.t.

\[
Y_1 + Y_2 + Y_3 \geq X_1 + X_2 + X_3 - Q
\]

\(0 \leq Y_1 \leq X_1, 0 \leq Y_2 \leq X_2, 0 \leq Y_3 \leq X_3\)
Define a separate recourse variable for each possible outcome:

\[ y_{ij} = \begin{array}{l}
\text{amount of shortfall in water delivered to user } i \text{ if } Q = q_j \\
\end{array} \]

In our "deterministic" LP formulation of the problem, then, we must simultaneously select the recourse (i.e., the user who will be denied the promised water) for each of the possible streamflows!

**EQUIVALENT DETERMINISTIC LP**

\[
\text{Max } 100X_1 + 50X_2 + 30X_3 - 0.2(250Y_1^1 + 75Y_2^1 + 60Y_3^1) \\
- 0.6(250Y_1^2 + 75Y_2^2 + 60Y_3^2) - 0.2(250Y_1^3 + 75Y_2^3 + 60Y_3^3)
\]

subject to

\[
\begin{array}{l}
X_1 + X_2 + X_3 - Y_1^1 - Y_2^1 - Y_3^1 \leq 4 \\
X_1 + X_2 + X_3 - Y_1^2 - Y_2^2 - Y_3^2 \leq 10 \\
X_1 + X_2 + X_3 - Y_1^3 - Y_2^3 - Y_3^3 \leq 17 \\
0 \leq Y_{1k}^k \leq X_1 \leq 2 \\
0 \leq Y_{2k}^k \leq X_2 \leq 3 \\
0 \leq Y_{3k}^k \leq X_3 \leq 5 \\
\forall k = 1, 2, 3 
\end{array}
\]
**Optimal Solution**

<table>
<thead>
<tr>
<th>Use</th>
<th>Allocation $X_i$</th>
<th>$Q=4$</th>
<th>10</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Municipal</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2 Industrial</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3 Agricultural</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Shortfall in Delivery $Y_i^1$ $Y_i^2$ $Y_i^3$

Objective value = $100(2) + 50(3) + 30(5) - 0.2[75(1) + 60(5)]$

= $500 - 0.2(375) = 425$

**Solution**

<table>
<thead>
<tr>
<th>Probability</th>
<th>0: neither bid is successful</th>
<th>1: bid #1 is successful, bid #2 is not</th>
<th>2: bid #2 is successful, bid #1 is not</th>
<th>3: both bids #1 and #2 are successful</th>
</tr>
</thead>
</table>

Probability

0.30

0.30

0.30

0.10

Possible Outcomes ("scenarios")

Par, Inc.

→
Stage 1 Variables

\[ X_1 = \# \text{ standard bags in the next quarter's prod'n plan} \]
\[ X_2 = \# \text{ deluxe bags in the next quarter's prod'n plan} \]

Stage 2 Variables

For outcome \# i (i=0,1,2,3)

\[ Y^i = \# \text{ standard bags added to next quarter's prod'n plan} \]
\[ T^i_{CD} = \text{ hours overtime in cut&dye} \]
\[ T^i_S = \text{ hours overtime in sewing} \]
\[ T^i_F = \text{ hours overtime in finishing} \]
\[ T^i_{IP} = \text{ hours overtime in inspect& pack} \]

Scenario #0: neither bid is successful

Second-stage problem (X is fixed)

Max \[ 8Y^0_1 - 5T^0_{CD} - 6T^0_S - 8T^0_F - 4T^0_{IP} \]

subject to

\[ \frac{7}{10}Y^0_1 - T^0_{CD} \leq 630 - \left[ \frac{7}{10}X_1 + X_2 \right] \]
\[ \frac{1}{2}Y^0_1 - T^0_S \leq 600 - \left[ \frac{1}{2}X_1 + \frac{5}{6}X_2 \right] \]
\[ Y^0_1 - T^0_F \leq 708 - \left[ X_1 + \frac{2}{3}X_2 \right] \]
\[ \frac{1}{10}Y^0_1 - T^0_{IP} \leq 135 - \left[ \frac{1}{10}X_1 + \frac{1}{4}X_2 \right] \]

\[ Y^0_1 \geq 0, T^0_{CD} \geq 0, T^0_S \geq 0, T^0_F \geq 0, T^0_{IP} \geq 0 \]
Scenario #1: only bid #1 is successful

Second-stage problem
(X is fixed)

Max \( 8Y_1^1 - 5T_{CD}^1 - 6T_S^1 - 8T_F^1 - 4T_{IP}^1 \)
subject to \( \frac{7}{10}Y_1^1 - T_{CD}^1 \leq 630 - 50 - \left[ \frac{7}{10}X_1 + X_2 \right] \)
\( \frac{1}{2}Y_1^1 - T_S^1 \leq 600 - 40 - \left[ \frac{1}{2}X_1 + \frac{5}{6}X_2 \right] \)
\( Y_1^1 - T_F^1 \leq 708 - 80 - \left[ X_1 + \frac{2}{3}X_2 \right] \)
\( \frac{1}{10}Y_1^1 - T_{IP}^1 \leq 135 - 10 - \left[ \frac{1}{10}X_1 + \frac{1}{4}X_2 \right] \)
\( Y_1^1 \geq 0, T_{CD}^1 \geq 0, T_S^1 \geq 0, T_F^1 \geq 0, T_{IP}^1 \geq 0 \)

Scenario #2: only bid #2 is successful

Second-stage problem
(X is fixed)

Max \( 8Y_1^2 - 5T_{CD}^2 - 6T_S^2 - 8T_F^2 - 4T_{IP}^2 \)
subject to \( \frac{7}{10}Y_1^2 - T_{CD}^2 \leq 630 - 30 - \left[ \frac{7}{10}X_1 + X_2 \right] \)
\( \frac{1}{2}Y_1^2 - T_S^2 \leq 600 - 50 - \left[ \frac{1}{2}X_1 + \frac{5}{6}X_2 \right] \)
\( Y_1^2 - T_F^2 \leq 708 - 70 - \left[ X_1 + \frac{2}{3}X_2 \right] \)
\( \frac{1}{10}Y_1^2 - T_{IP}^2 \leq 135 - 15 - \left[ \frac{1}{10}X_1 + \frac{1}{4}X_2 \right] \)
\( Y_1^2 \geq 0, T_{CD}^2 \geq 0, T_S^2 \geq 0, 100 \geq T_F^2 \geq 0, T_{IP}^2 \geq 0 \)
Scenario #3: both bids are successful

Second-stage problem

\( X \) is fixed

Max \( 8Y_1^3 - 5T_{CD}^3 - 6T_S^3 - 8T_F^3 - 4T_{IP}^3 \)

subject to

\[
\begin{align*}
\frac{7}{10}Y_1^3 - T_{CD}^3 &\leq 630 - 50 - 30 - [\frac{7}{10}X_1 + X_2] \\
\frac{1}{2}Y_1^3 - T_S^3 &\leq 600 - 40 - 50 - [\frac{1}{2}X_1 + \frac{5}{6}X_2] \\
Y_1^3 - T_F^3 &\leq 708 - 80 - 70 - [X_1 + \frac{2}{3}X_2] \\
\frac{1}{10}Y_1^3 - T_{IP}^3 &\leq 135 - 10 - 15 - [\frac{1}{10}X_1 + \frac{1}{4}X_2] \\
Y_1^3 &\geq 0, T_{CD}^3 \geq 0, T_S^3 \geq 0, T_F^3 \geq 0, T_{IP}^3 \geq 0
\end{align*}
\]

Objective

Max \( 10X_1 + 9X_2 + 0.3 \left( 8Y_1^0 - 5T_{CD}^0 - 6T_S^0 - 8T_F^0 - 4T_{IP}^0 \right) \)

\[
\begin{align*}
&+ 0.3 \left( 8Y_1^1 - 5T_{CD}^1 - 6T_S^1 - 8T_F^1 - 4T_{IP}^1 \right) \\
&+ 0.3 \left( 8Y_1^2 - 5T_{CD}^2 - 6T_S^2 - 8T_F^2 - 4T_{IP}^2 \right) \\
&+ 0.1 \left( 8Y_1^3 - 5T_{CD}^3 - 6T_S^3 - 8T_F^3 - 4T_{IP}^3 \right)
\end{align*}
\]

Equivalent Deterministic Linear Programming Model
subject to

scenario #0

\[
\begin{align*}
\frac{7}{10}x_1 + x_2 + \frac{7}{10}y_1 - T_{CD}^0 & \leq 630 \\
\frac{1}{2}x_1 + \frac{5}{6}x_2 + \frac{1}{2}y_1 - T_S^0 & \leq 600 \\
x_1 + \frac{2}{3}x_2 + y_1 - T_F^0 & \leq 708 \\
\frac{1}{10}x_1 + \frac{1}{4}x_2 + \frac{1}{10}y_1 - T_{IP}^0 & \leq 135 \\
T_F^0 & \leq 100
\end{align*}
\]

scenario #1

\[
\begin{align*}
\frac{7}{10}x_1 + x_2 + \frac{7}{10}y_1 - T_{CD}^1 & \leq 580 \\
\frac{1}{2}x_1 + \frac{5}{6}x_2 + \frac{1}{2}y_1 - T_S^1 & \leq 560 \\
x_1 + \frac{2}{3}x_2 + y_1 - T_F^1 & \leq 628 \\
\frac{1}{10}x_1 + \frac{1}{4}x_2 + \frac{1}{10}y_1 - T_{IP}^1 & \leq 125 \\
T_F^1 & \leq 100
\end{align*}
\]

scenario #2

\[
\begin{align*}
\frac{7}{10}x_1 + x_2 + \frac{7}{10}y_2 - T_{CD}^2 & \leq 600 \\
\frac{1}{2}x_1 + \frac{5}{6}x_2 + \frac{1}{2}y_2 - T_S^2 & \leq 550 \\
x_1 + \frac{2}{3}x_2 + y_2 - T_F^2 & \leq 638 \\
\frac{1}{10}x_1 + \frac{1}{4}x_2 + \frac{1}{10}y_2 - T_{IP}^2 & \leq 120 \\
T_F^2 & \leq 100
\end{align*}
\]

scenario #3

\[
\begin{align*}
\frac{7}{10}x_1 + x_2 + \frac{7}{10}y_3 - T_{CD}^3 & \leq 550 \\
\frac{1}{2}x_1 + \frac{5}{6}x_2 + \frac{1}{2}y_3 - T_S^3 & \leq 510 \\
x_1 + \frac{2}{3}x_2 + y_3 - T_F^3 & \leq 558 \\
\frac{1}{10}x_1 + \frac{1}{4}x_2 + \frac{1}{10}y_3 - T_{IP}^3 & \leq 110 \\
T_F^3 & \leq 100
\end{align*}
\]

\[
\begin{align*}
x_1 & \geq 0, \ x_2 & \geq 0, \ y_1^i & \geq 0, \ T_{CD}^i & \geq 0, \\
T_{CD}^i & \geq 0, \ T_S^i & \geq 0, \ T_F^i & \geq 0, \ T_{IP}^i & \geq 0 \\
i = & 0, 1, 2, 3
\end{align*}
\]