Solving
Linear Equations

author

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Elementary Row Operations

- Multiply any row of the matrix by a (positive or negative) scalar
- Add to any row a scalar multiple of another row
- Interchange two rows of the matrix

(Strictly speaking, the third is not "elementary", because it can be accomplished by a sequence of the other two row operations!)

⏎

Elementary Row Operations

- Multiply any row of the matrix by a (positive or negative) scalar

APL

```
   ∇B ← ki   ERO1 A;k;i
[1] k←kr[1]
[2] i←kr[2]
[3] B ← A
[4] B[i;] ← k × B[i;]
   ∇
```

Elementary Row Operations

- Add to any row a scalar multiple of another row

APL

```
   ∇B ← kij  ERO2 A;k;i;j
[1] k←kij [1]
[2] i←kij [2]
[3] j←kij [3]
[4] B ← A
[5] B[i;] ← B[i;]+k × B[j;]
   ∇
```

Elementary Row Operations

- Interchange two rows of the matrix

APL

```
   ∇B ← ij  ERO2 A;i;j
[1] i← ij[1]
[2] j← ij[2]
[3] B ← A
[4] B[i ;] ← B[j ;]
[5] B[j ;] ← A[i ;]
   ∇
```

Elementary Column Operations

- Multiply any column by a (positive or negative) scalar
- Add to any column a scalar multiple of another column
- Interchange two columns of the matrix

Equivalence of Matrices

Matrix **A** is ***equivalent*** to matrix **B** (**A** ~ **B**) if **B** is the result of a sequence of elementary row &/or column operations on **A**.

If only row operations are used, then **A** is ***row-equivalent*** to B

If only column operations are used, then **A** is ***column-equivalent*** to B

Echelon Matrix

--an mxn matrix with the properties

- each of the first k ($0 \le k \le m$) rows has some nonzero entries, and the remaining m-k rows consist only of zeroes
- the first nonzero entry in each of the first k rows is a "1"
- in each of the first k rows, the number of zeroes preceding the leading "1" is smaller than it is in the next row

ECHELON MATRIX

Example

$$\begin{bmatrix} 1 & 5 & 0 & 3 & -1 & 2 & 8 \\ 0 & 0 & 1 & -1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} k=3 \quad = \textit{rank} \\ \\ \\ \end{matrix}$$

Note:* *every matrix is row-equivalent to some echelon matrix.

Theorem

If A is equivalent to B, then the rank of A equals the rank of B.

RANK: size of the largest (square) nonsingular submatrix

Elementary Matrices

An ***elementary matrix*** E is the result of performing an elementary operation on an identity matrix.

Example

(Elementary row operation: add −2 times first row to third row)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Multiplication by an Elementary Matrix

pre-multiplication by elementary matrix

If E is an mxm elementary matrix and A is an mxn matrix, then EA equals the result of performing the same elementary ***row*** operation on matrix A.

Example:

add −2 times first row to third row

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 4 \\ 5 & 1 & 3 & -1 \\ 4 & 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 5 & 1 & 3 & -1 \\ 0 & 5 & 1 & -6 \end{bmatrix}$$

If E is an mxm elementary matrix and A is an mxn matrix, then AE equals the result of performing the same elementary ***column*** operation on matrix A.

Example:

add –2 times third ***column*** *to first* ***column***

$$\begin{bmatrix}2&-1&0\\5&1&3\\4&3&1\\0&2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}=\begin{bmatrix}2&-1&0\\-1&1&3\\2&3&1\\-2&2&1\end{bmatrix}$$

result of subtracting twice third column from first

post–multiplication by elementary matrix

Calculation of Matrix Inverse

To compute A^{-1}, augment the matrix A on the right by the appropriate identity matrix $\boxed{A \mid I}$, and perform elementary row operations on this matrix to obtain $\boxed{I \mid P}$. Then $P = A^{-1}$

Calculation of Matrix Inverse

Example:

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -4 & -5 & 3 \\ 0 & 1 & 0 & 3 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

and so $$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

Pivot

Pivot operation on row **r**, column **s**
i.e., element A_r^s of mxn matrix A:

A sequence of elementary row operations:

- For i=1,2,...m but i≠r:
  add $-A_i^s/A_r^s$ times row r to row i
- Multiply row r by the scalar $1/A_r^s$

Effect: column s will consist of zeroes, with the exception of a "1" in row r.

Warning: this is not the only sequence of elementary row operations having this effect!

Pivot

$$\begin{tikzcd} & \begin{bmatrix} 1 & \frac{3}{5} & 0 \\ -1 & -\frac{4}{3} & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & \textcircled{3} \end{bmatrix} \arrow[ur] \arrow[dr] & \\ & \begin{bmatrix} 2 & 3 & 0 \\ -1 & -\frac{4}{3} & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} \end{tikzcd}$$

A pivot!

$R_1 \leftarrow R_1 - \frac{1}{3} R_3$

$R_2 \leftarrow R_2 - \frac{1}{3} R_3$

$R_3 \leftarrow \frac{1}{3} R_3$

***Not** a pivot!*

$R_1 \leftarrow R_1 - R_2$

$R_2 \leftarrow R_2 - \frac{1}{3} R_3$

$R_3 \leftarrow \frac{1}{3} R_3$

Pivot Matrix

A pivot matrix corresponding to a pivot on row r, column s of a matrix A is the result of performing the same elementary row operations on the mxm identity matrix.

A pivot matrix is the product of elementary matrices!

Pivot Matrix

Differs from the mxm identity matrix only in column r

$$\begin{bmatrix} 1 & 0 & \cdots & -\frac{A_1^s}{A_r^s} & \cdots & 0 & 0 \\ 0 & 1 & \cdots & -\frac{A_2^s}{A_r^s} & \cdots & 0 & 0 \\ & & \ddots & & & & \\ 0 & 0 & \cdots & \frac{1}{A_r^s} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -\frac{A_{m-1}^s}{A_r^s} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -\frac{A_m^s}{A_r^s} & \cdots & 0 & 1 \end{bmatrix}$$

Pivot Matrix

```
  ∇ P ← ij  PIVOTMATRIX  A;i;j;M
[1] i ← ij[1]
[2] j ← ij[2]
[3] P ← IDENTITY  M←(ρA)[1]
[4] P[(i≠ιM)/ιM;i]←(i≠ιM)/-A[;j]÷A[i;j]
```

APL

Pivot Matrix

To store a pivot matrix, we need not store the entire matrix, but only

- the number (r) of the pivot row
- column #r of the pivot matrix (the ***eta*** vector)

$$\eta = \left[-\frac{A_1^s}{A_r^s}, -\frac{A_2^s}{A_r^s}, \cdots \frac{1}{A_r^s}, \cdots -\frac{A_m^s}{A_r^s} \right]$$

This is sufficient information to reconstruct the pivot matrix.

Product Form of the Inverse

If matrix A is nonsingular, then a sequence of pivots down the diagonal of A (with possible row interchanges to avoid zero pivot elements) will reduce A to the identity matrix. This is equivalent to pre-multiplying A by a sequence of pivot matrices:

$$(P_m \cdots (P_3(P_2(P_1A)\,)\,)\cdots\,) = I$$
$$\Longrightarrow \quad (P_m \cdots\; P_3\, P_2\, P_1)\, A = I$$
$$\Longrightarrow \quad A^{-1} = P_m \cdots\; P_3\, P_2\, P_1$$

Product Form of the Inverse

In the Revised Simplex Method, computation of values in the tableau is done, not by pivoting in the tableau, but by either pre-multiplication or post-multiplication by the inverse matrix:

- Computation of simplex multipliers

$$\pi = \mathbf{c}^B(\mathbf{A}^B)^{-1}$$

used in selecting pivot column

- Computation of substitution rates

$$\alpha = (\mathbf{A}^B)^{-1}\mathbf{A}^s$$

used in performing the pivot

Computing Simplex Multipliers

Solve $\pi\ \mathbf{A}^B = \mathbf{c}^B$ for π :

$$\begin{aligned}\pi &= \mathbf{c}^B(\mathbf{A}^B)^{-1}\\ &= \mathbf{c}^B\ (\mathrm{P}_k\ \mathrm{P}_{k-1}\cdots\mathrm{P}_3\mathrm{P}_2\mathrm{P}_1)\\ &= (((\cdots(\mathbf{c}^B\mathrm{P}_k)\mathrm{P}_{k-1}\cdots\mathrm{P}_3)\mathrm{P}_2)\mathrm{P}_1)\end{aligned}$$

"Backward Transformation", or BTRAN

The pivot matrices are processed in the ***reverse*** of the order in which they were generated, i.e., $P_k\ P_{k-1}\ldots\ P_3\,P_2\,P_1$

BTRAN

For each pivot matrix P, we need to calculate $\pi = \nu P$

column r

$$\pi = \begin{bmatrix} \nu_1 & \nu_2 & \cdots & \nu_{m-1} & \nu_m \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & \eta_1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \eta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_r & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_{m-1} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \eta_m & \cdots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \nu_1 & \nu_2 & \cdots & \left(\sum_i \nu_i \eta_i\right) & \cdots & \nu_{m-1} & \nu_m \end{bmatrix}$$

entry r

BTRAN

$$\pi_j = \begin{cases} \nu_j & \textbf{for } \mathbf{j \neq r} \\ \sum_i \nu_i \eta_i & \textbf{for } \mathbf{j = r} \end{cases}$$

Step 0: Set $\mathbf{v} = \mathbf{c}^B$ and k = # of ETA vectors

Step 1: Using BTRAN formula above, compute with ETA vector #k

Step 2: If k>1, let $\mathbf{v} = \boldsymbol{\pi}$ and k = k−1, and go to step 1; else proceed to step 3.

Step 3: The final value of $\boldsymbol{\pi}$ is the solution of $\boldsymbol{\pi}\ \mathbf{A}^B = \mathbf{c}^B$

FTRAN

Solve $A^B \alpha = A^s$ for substitution rates α

$$\begin{aligned}\alpha &= (A^B)^{-1} A^s \\ &= (P_k\, P_{k-1} \cdots P_3 P_2 P_1) A^s \\ &= (P_k(P_{k-1} \cdots P_3(P_2(P_1 A^s)) \cdots))\end{aligned}$$

"Forward Transformation", or FTRAN

The pivot matrices are processed in the same order that they were generated,

i.e., $P_1, P_2, P_3, \cdots \quad P_{k-1}, P_k$

FTRAN

$$\alpha = \begin{array}{c} \begin{array}{ccccccc} & & & \text{column } r \downarrow & & & \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & \eta_1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \eta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_r & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_{m-1} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \eta_m & \cdots & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_m \end{bmatrix} = \begin{bmatrix} \nu_1 + \eta_1 \nu_r \\ \nu_2 + \eta_2 \nu_r \\ \vdots \\ \eta_r \nu_r \\ \nu_m + \eta_m \nu_r \end{bmatrix}$$

That is,

$$\alpha_i = \begin{cases} \nu_i + \eta_i \nu_r & \textbf{for } \mathbf{i \neq r} \\ \eta_r \nu_r & \textbf{for } \mathbf{i = r} \end{cases}$$

FTRAN

$$\alpha_i = \begin{cases} \nu_i + \eta_i \nu_r & \textbf{for } \mathbf{i \neq r} \\ \eta_r \nu_r & \textbf{for } \mathbf{i = r} \end{cases}$$

Step 0: Set $\boldsymbol{\nu} = A^s$ (e.g., column of original tableau), and k=1.

Step 1: Using the FTRAN formula above, compute α

Step 2: If k < # of ETA vectors, then let $\boldsymbol{\nu} = \alpha$ and k=k+1, and go to step 1; else proceed to step 3.

Step 3: The final value of $\boldsymbol{\nu}$ is the solution α of the equation $\mathbf{A}^B \alpha = \mathbf{A}^s$

Gauss Elimination

-- a method for solving Ax=b by performing a sequence of elementary row operations on the augmented matrix $\boxed{A \mid b}$ to reduce it to an echelon matrix. The solution is then obtained by "back-substitution".

Example: $\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 + 2x_3 = 2 \\ -x_1 - x_2 + x_3 = 2 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 2 \\ -1 & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Longrightarrow \begin{cases} x_1 + x_2 + x_3 = 4 \\ \qquad x_2 + x_3 = -2 \\ \qquad\qquad x_3 = 3 \end{cases}$$

Backsubstitution:

$$\left\{\begin{array}{l} x_1 = 4 - x_2 - x_3 \\ \left.\begin{array}{l} x_2 = -2 - x_3 \\ x_3 = 3 \end{array}\right\} \Longrightarrow x_2 = -5 \end{array}\right\} \Longrightarrow x_1 = 6$$

Gauss-Jordan Elimination

--similar to Gauss elimination, except that the coefficient matrix is diagonalized by further elementary row operations, eliminating non-zeroes above as well as below the diagonal. Eliminates the need for "back-substitution".

Example: $\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 + 2x_3 = 2 \\ -x_1 - x_2 + x_3 = 2 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 2 \\ -1 & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

That is, $\begin{cases} x_1 = 6 \\ x_2 = -5 \\ x_3 = 3 \end{cases}$

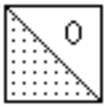
Compared to "Gauss Elimination Plus Back Substitution", Gauss-Jordan Elimination requires more computation--
especially if the equations are to be solved for several right-hand-side vectors!

Gauss Elimination as Matrix Factorization

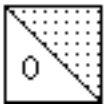
$$A = P\,L\,U$$

P is a permutation matrix (which performs the interchange of rows for partial pivoting)

L is a lower triangular matrix,



U is an upper triangular matrix



$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_2 + R_1]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_2]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Upper-triangular matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Lower triangular matrices

$$\underbrace{E_2 E_1}_{\hat{L}} A = U$$

Lower triangular matrix

$$\hat{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad \hat{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = L$$

$$\hat{L}\,A = U \implies A = \hat{L}^{-1}U = L\,U$$

Matrix A is factored into a product of lower & upper triangular matrices!

Suppose that we need to solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ -x_1 \quad - x_2 + x_3 = 5 \\ \qquad x_2 + 3x_3 = -1 \end{cases}$$

$$A = L\,U = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

To solve Ax=b, i.e., L(Ux)=b:

- solve Ly=b for y *(forward substitution)*

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&1&1\end{bmatrix}\mathbf{y}=\begin{bmatrix}2\\5\\-1\end{bmatrix}\Longrightarrow\begin{cases}\mathbf{y}_1=\mathbf{2}\\ \mathbf{y}_2=\mathbf{5}+\mathbf{y}_1=\mathbf{7}\\ \mathbf{y}_3=\mathbf{-1}-\mathbf{y}_2=\mathbf{-8}\end{cases}$$

- solve Ux=y for x *(backward substitution)*

$$\begin{bmatrix}1&2&1\\0&1&2\\0&0&1\end{bmatrix}\mathbf{x}=\begin{bmatrix}2\\7\\-8\end{bmatrix}\Longrightarrow\begin{cases}\mathbf{x}_1=\mathbf{2}-\mathbf{2x}_2-\mathbf{x}_3=\mathbf{-36}\\ \mathbf{x}_2=\mathbf{7}-\mathbf{2x}_3=\mathbf{23}\\ \mathbf{x}_3=-\mathbf{8}\end{cases}$$

LDL^T Factorization

Of primary interest when the matrix A is symmetric and positive definite, e.g., Hessian of a convex function.

Start with the factorization: $\mathbf{A}=\mathbf{LU}$

Let D be a diagonal matrix with $\mathbf{D}_i^i=\mathbf{U}_i^i$

Then $\mathbf{D}^{-1}$ is a diagonal matrix with elements $1/\mathbf{D}_i^i$

Define $\widehat{\mathbf{U}}=\mathbf{D}^{-1}\mathbf{U}$ $\Rightarrow$ $\mathbf{U}=\mathbf{D}\widehat{\mathbf{U}}$ so that $\mathbf{A}=\mathbf{LD}\widehat{\mathbf{U}}$

By symmetry of A, $\mathbf{A}=\mathbf{A}^T$

$$\Rightarrow\mathbf{LD}\widehat{\mathbf{U}}=(\mathbf{LD}\widehat{\mathbf{U}})^T=\widehat{\mathbf{U}}^T\mathbf{D}^T\mathbf{L}^T$$

$$\Rightarrow\widehat{\mathbf{U}}=\mathbf{L}^T$$

That is, $\mathbf{A}=\mathbf{LDL}^T$

L is lower triangular, and D is diagonal

Suppose that $\mathbf{A} = \mathbf{LDL^T}$

Consider the quadratic form $\mathbf{x^T A x} = \sum_i^n \sum_j^n \mathbf{A}_i^j \mathbf{x}_i \mathbf{x}_j$

$$\mathbf{x^T A x} = \mathbf{x^T LDL^T x} = [\mathbf{L^T x}]^T \mathbf{D} [\mathbf{L^T x}] = \mathbf{y^T D y} = \sum_i^n \mathbf{D}_i^i \mathbf{y}_i^2$$

where $\mathbf{y} = \mathbf{L^T x}$

If $\mathbf{D}_i^i \geq 0$, then, $\mathbf{x^T A x} \geq 0$ for all x

A is positive semidefinite

If $\mathbf{D}_i^i > 0$, $\mathbf{x^T A x} > 0$ for all $x \neq 0$ ($\Longrightarrow y \neq 0$)

A is positive definite

Cholesky Factorization

symmetric

$$\mathbf{A} = \hat{\mathbf{L}}\, \hat{\mathbf{L}}^T$$

Lower triangular

Suppose that we have the factorization $\mathbf{A} = \mathbf{LDL^T}$

Define a new diagonal matrix $\hat{\mathbf{D}}$ where $\hat{\mathbf{D}}_i^i = \sqrt{\mathbf{D}_i^i}$

so that $\mathbf{D} = \hat{\mathbf{D}}\hat{\mathbf{D}}$

Then $\mathbf{A} = \mathbf{LDL^T} = \mathbf{L}\hat{\mathbf{D}}\hat{\mathbf{D}}\mathbf{L^T} = \mathbf{L}\hat{\mathbf{D}}\hat{\mathbf{D}}^T\mathbf{L^T} = \mathbf{L}\hat{\mathbf{D}}\,[\mathbf{L}\hat{\mathbf{D}}]^T$

$= \hat{\mathbf{L}}\hat{\mathbf{L}}^T$ where $\hat{\mathbf{L}} = \mathbf{L}\hat{\mathbf{D}}$

Example

We wish to find the Cholesky factorization of the matrix

$$A=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow[\text{Inverse: } R_3 \leftarrow R_3 + \frac{1}{2} R_1]{R_3 \leftarrow R_3 - \frac{1}{2} R_1} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ -\frac{1}{2} & 0 & 1 & 0 & 1 & \frac{3}{2} \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{matrix}}_{L^{-1}\text{ (lower triangular)}} \;\middle|\; \underbrace{\begin{matrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \end{matrix}}_{U\text{ (upper triangular)}} \right]$$

Inverse: $R_3 \leftarrow R_3 + R_2$

L is found by performing (on the identity matrix) the inverse of the row operations used to reduce the A matrix:

$$\begin{matrix} R_3 \leftarrow R_3 + \frac{1}{2} R_1 \\ R_3 \leftarrow R_3 + R_2 \end{matrix} \quad \Longrightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}$$

We now have the LU factorization of matrix A:

$$\mathbf{A} = \mathbf{L}\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1/2 \end{bmatrix}$$

Define the diagonal matrix D:

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \Longrightarrow \mathbf{D}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Diagonal elements of matrix **U**

Reciprocals of diagonal elements of D

Note that $\widehat{\mathbf{U}} = \mathbf{D}^{-1}\,\mathbf{U} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{L}^{T}$$

And so,

$$\mathbf{A} = \mathbf{L}\mathbf{D}\,\mathbf{L}^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Cholesky Factorization

Define the diagonal matrix $\widehat{D}$ where $\widehat{D}^i_i = \sqrt{D^i_i}$

$$\widehat{D} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$\widehat{L} = L\widehat{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 1 & 1/\sqrt{2} \end{bmatrix}$$

$$A = \widehat{L}\widehat{L}^T = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 1 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2}/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

This is the Cholesky factorization of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$