Simplex Algorithm:

Pivoting in the Tableau

$$\begin{aligned} \text{Maximize} \quad & 12X_1 + 8X_2 \\ \text{subject to} \quad & 5X_1 + 2X_2 \leq 150 \\ & 2X_1 + 3X_2 \leq 100 \\ & 4X_1 + 2X_2 \leq 80 \\ & X_1 \geq 0, X_2 \geq 0 \end{aligned}$$

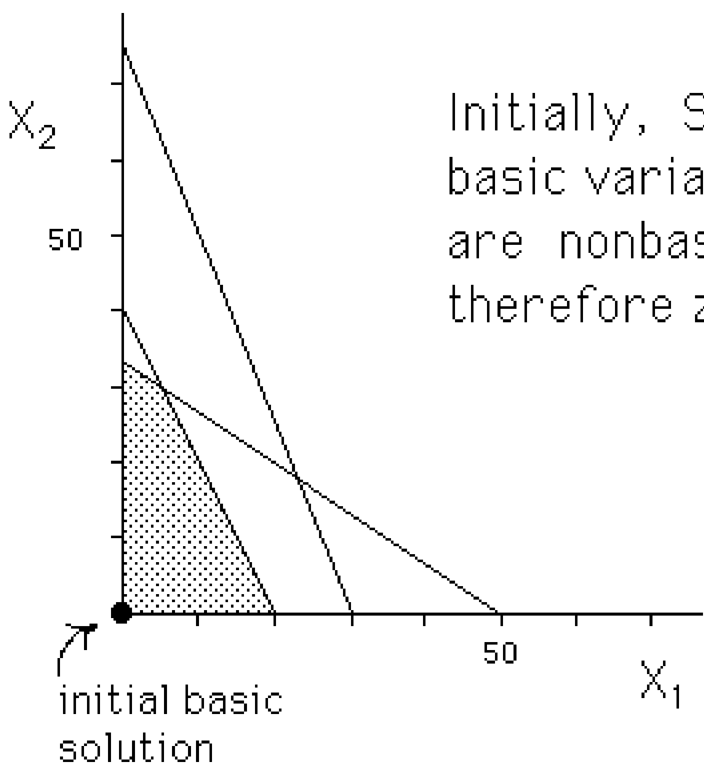
Inequalities are converted to equations by the introduction of slack variables.

$$\begin{aligned} \text{Maximize} \quad & 12X_1 + 8X_2 && = Z \\ \text{subject to} \quad & 5X_1 + 2X_2 + S_1 && = 150 \\ & 2X_1 + 3X_2 + S_2 && = 100 \\ & 4X_1 + 2X_2 + S_3 && = 80 \\ & X_1 \geq 0, X_2 \geq 0, S_1 \geq 0\ S_2 \geq 0, S_3 \geq 0 \end{aligned}$$

$$\begin{aligned}
\text{Maximize}\quad & 12X_1 + 8X_2 = Z \\
\text{subject to}\quad & 5X_1 + 2X_2 + S_1 = 150 \\
& 2X_1 + 3X_2 + S_2 = 100 \\
& 4X_1 + 2X_2 + S_3 = 80 \\
& X_1 \geq 0, X_2 \geq 0, S_1 \geq 0\ S_2 \geq 0, S_3 \geq 0
\end{aligned}$$

Tableau

-Z	X_1	X_2	S_1	S_2	S_3	rhs
1	12	8	0	0	0	0
0	5	2	1	0	0	150
0	2	3	0	1	0	100
0	4	2	0	0	1	80



Initially, S_1, S_2, & S_3 are basic variables, while X_1 & X_2 are nonbasic variables (and therefore zero!)

$-Z$	X_1	X_2	S_1	S_2	S_3	rhs
1	12	8	0	0	0	0
0	5	2	1	0	0	150
0	2	3	0	1	0	100
0	4	2	0	0	1	80

Increasing either X_1 or X_2 will increase the objective function. Let's choose X_1.

Minimum ratio test:

$$\text{Min}\left\{\frac{150}{5}, \frac{100}{2}, \frac{80}{4}\right\}$$

$$= \text{Min}\{30, 50, 20\} = 20$$

Minimum ratio test:

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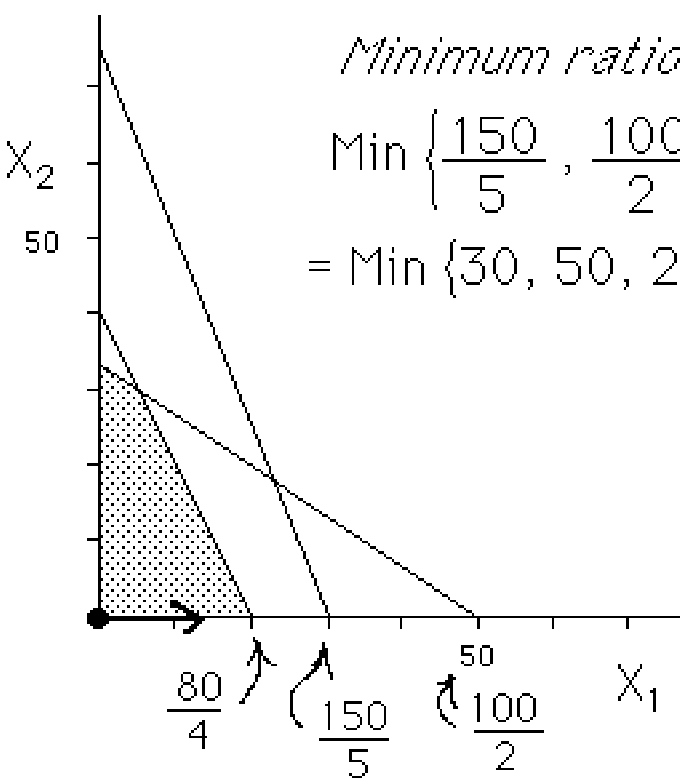
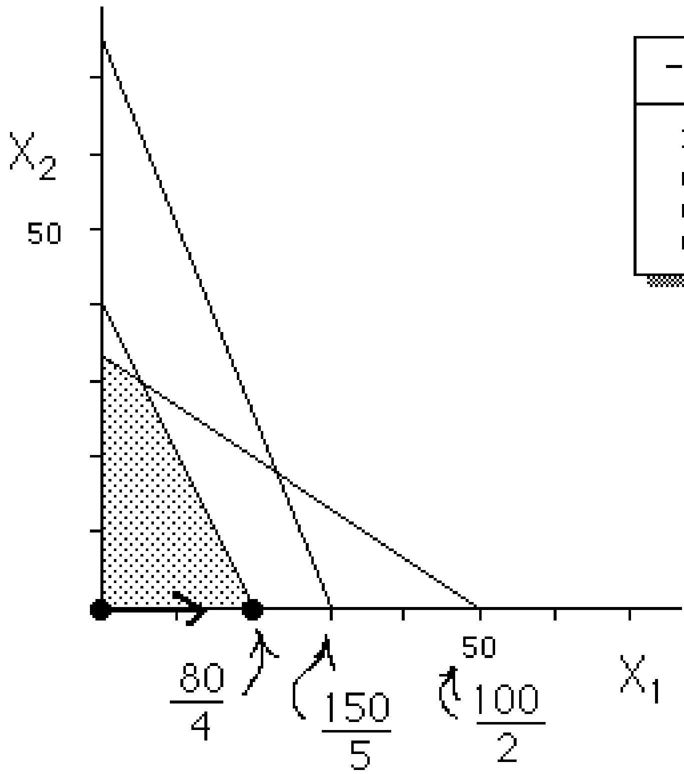




-Z	X_1	X_2	S_1	S_2	S_3	rhs
1	0	2	0	0	-3	-240
0	0	-0.5	1	0	-1.25	50
0	0	2	0	1	-0.5	60
0	1	0.5	0	0	0.25	20

Pivoting has taken us from one "corner" to an adjacent "corner"

-Z	X_1	X_2	S_1	S_2	S_3	rhs
1	0	2	0	0	-3	-240
0	0	-0.5	1	0	-1.25	50
0	0	(2)	0	1	-0.5	60
0	1	0.5	0	0	0.25	20

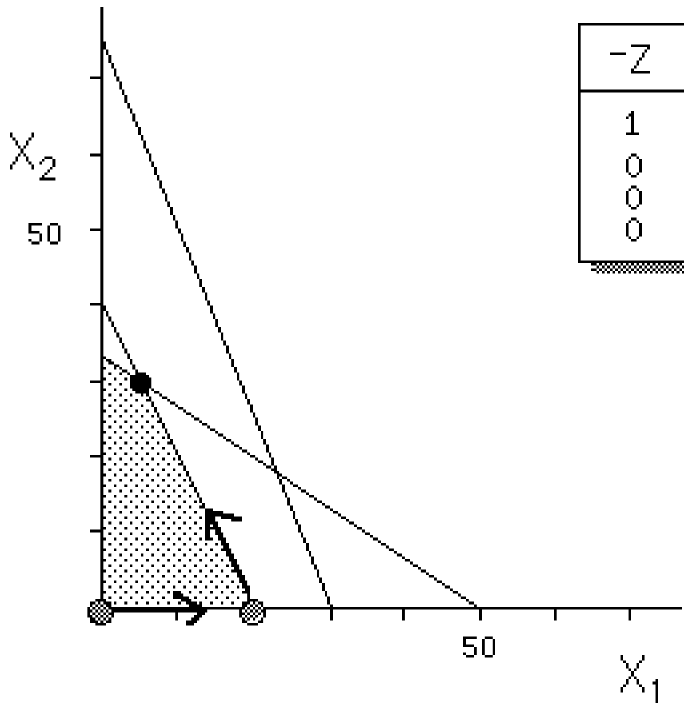
The objective function may be increased further by increasing X_2 .

Minimum ratio test

$$\text{Min}\left\{ —, \frac{60}{2}, \frac{20}{0.5} \right\}$$

$$= \text{Min}\{—, 30, 40\} = 30$$

(Only ratios of RHS's to positive substitution rates are used in the minimum ratio test!)

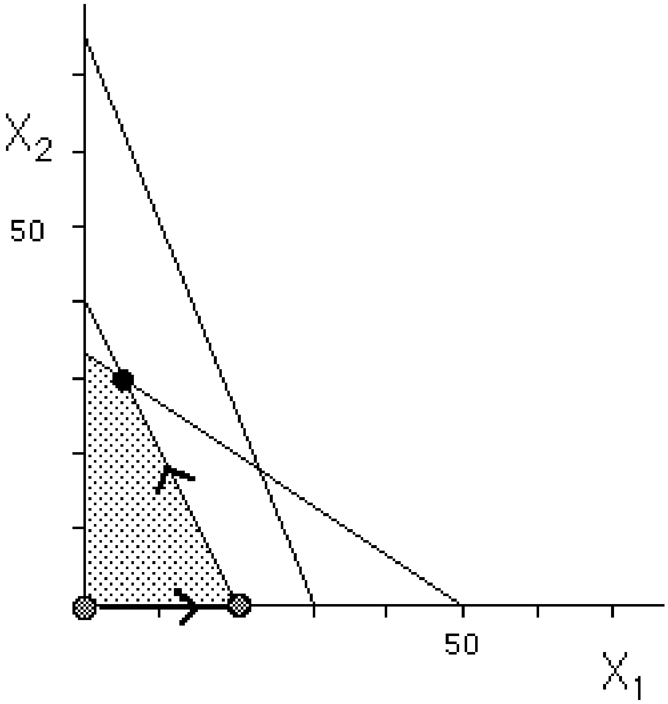


−Z	X_1	X_2	S_1	S_2	S_3	rhs
1	0	0	0	−1	−2.5	−300
0	0	0	1	0.25	−1.38	65
0	0	1	0	0.5	−0.25	30
0	1	0	0	−0.25	0.375	5

−Z	X_1	X_2	S_1	S_2	S_3	rhs
1	0	0	0	−1	−2.5	−300
0	0	0	1	0.25	−1.38	65
0	0	1	0	0.5	−0.25	30
0	1	0	0	−0.25	0.375	5

Because no variable has a positive relative profit, this basic solution is optimal, i.e.,

$$\begin{cases} Z = 300 \\ S_1 = 65 \\ X_2 = 30 \\ X_1 = 5 \end{cases}$$



-Z	X1	X2	S1	S2	S3	rhs
1	12	8	0	0	0	0
0	5	2	1	0	0	150
0	2	3	0	1	0	100
0	4	2	0	0	1	80

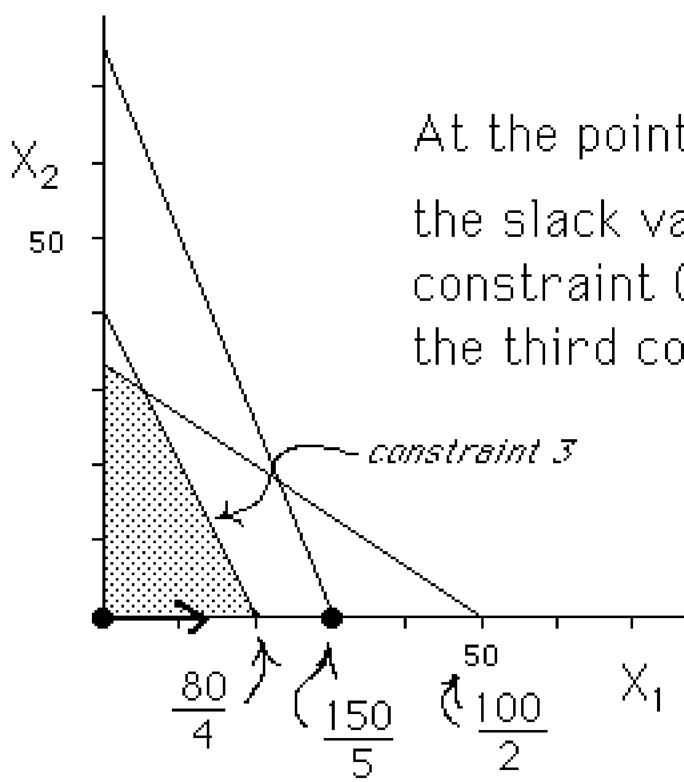
What if you make a mistake in performing the "minimum-ratio test"?

Suppose, in the first tableau, we had mistakenly chosen to pivot in row 2 (rather than in the correct row, which is row 4).

$-Z$	X_1	X_2	S_1	S_2	S_3	rhs
1	12	8	0	0	0	0
0	(5)	2	1	0	0	150
0	2	3	0	1	0	100
0	4	2	0	0	1	80

$-Z$	X_1	X_2	S_1	S_2	S_3	rhs
1	0	3.2	-2.4	0	0	-360
0	1	0.4	0.2	0	0	30
0	0	2.2	-0.4	1	0	40
0	0	0.4	-0.8	0	1	-40

The result is a tableau with a negative basic variable!



At the point $X_1 = 0$, $X_2 = 30$,

the slack variable for the third constraint (S_3) is negative, i.e., the third constraint is violated!

Maximize $3X_1 + 2X_2$
subject to $6X_1 + 4X_2 \le 24$
$10X_1 + 3X_2 \le 30$
$X_1 \ge 0, X_2 \ge 0$

Example: Multiple Optima

Maximize $3X_1 + 2X_2 = Z$
subject to $6X_1 + 4X_2 + S_1 = 24$
$10X_1 + 3X_2 + S_2 = 30$
$X_1 \ge 0, X_2 \ge 0, S_1 \ge 0, S_2 \ge 0$

Maximize $3X_1 + 2X_2 = Z$
subject to $6X_1 + 4X_2 + S_1 = 24$
$10X_1 + 3X_2 + S_2 = 30$
$X_1 \ge 0, X_2 \ge 0, S_1 \ge 0, S_2 \ge 0$

$-Z$	X_1	X_2	S_1	S_2	rhs
1	3	2	0	0	0
0	6	4	1	0	24
0	10	3	0	1	30

-Z	X_1	X_2	S_1	S_2	rhs
1	3	2	0	0	0
0	6	4	1	0	24
0	10	3	0	1	30

-Z	X_1	X_2	S_1	S_2	rhs
1	0	1.1	0	-0.3	-9
0	0	2.2	1	-0.6	6
0	1	0.3	0	0.1	3

-Z	X_1	X_2	S_1	S_2	rhs
1	0	1.1	0	-0.3	-9
0	0	2.2	1	-0.6	6
0	1	0.3	0	0.1	3

-Z	X_1	X_2	S_1	S_2	rhs
1	0	0	-0.5	0	-12
0	0	1	0.455	-0.273	2.73
0	1	0	-0.136	0.182	2.18

There is no positive relative profit, so this basic solution is optimal.

Note, however, that there is a nonbasic variable (S_2) with a ***zero*** *relative profit.*

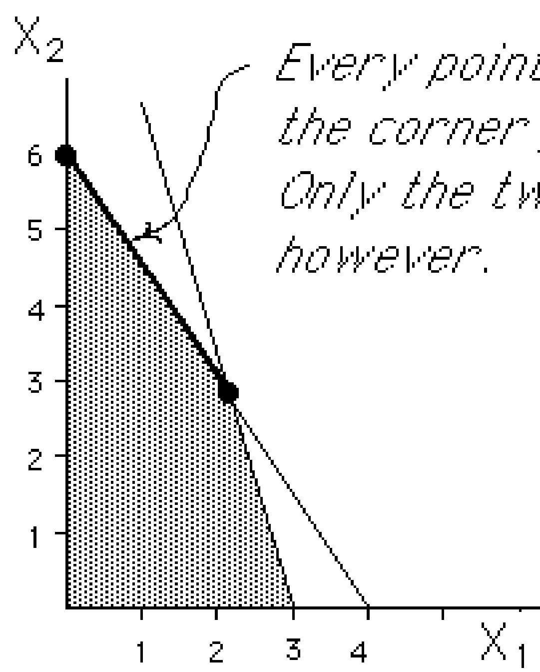
-Z	X_1	X_2	S_1	S_2	rhs
1	0	0	⁻0.5	0	⁻12
0	0	1	0.455	⁻0.273	2.73
0	1	0	⁻0.136	0.182	2.18

basic solution: $Z = 12$, $X_2 = 2.73$, $X_1 = 2.18$

-Z	X_1	X_2	S_1	S_2	rhs
1	0	0	⁻0.5	0	⁻12
0	1.5	1	0.25	0	6
0	5.5	0	⁻0.75	1	12

basic solution: $Z = 12$, $X_2 = 6$, $S_2 = 12$

Both of these basic solutions are optimal!



Every point on this edge (including the corner points) is optimal. Only the two corners are basic, however.

Example: Unbounded Solution

Maximize $2X_1 + 3X_2$
subject to $X_1 + X_2 \geq 3$
$X_1 - 2X_2 \leq 4$
$X_1 \geq 0, X_2 \geq 0$

Maximize $2X_1 + 3X_2$
subject to $X_1 + X_2 - S_1 = 3$
$X_1 - 2X_2 + S_2 = 4$
$X_1 \geq 0, X_2 \geq 0, S_1 \geq 0, S_2 \geq 0$

-Z	X_1	X_2	S_1	S_2	rhs
1	-1	0	3	0	-9
0	1	1	-1	0	3
0	3	0	-2	1	10

Suppose that, after one or more pivots, we obtain this tableau.

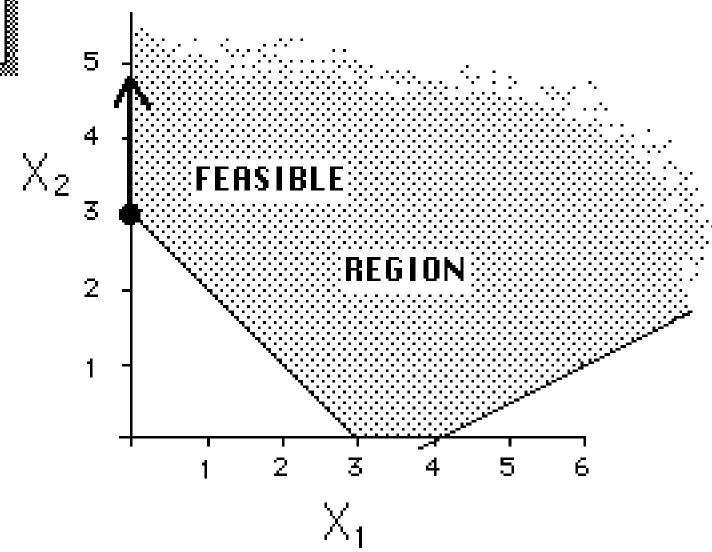
This tableau is not optimal, because S_1 has a positive relative profit.

-Z	X_1	X_2	S_1	S_2	rhs
1	-1	0	3	0	-9
0	1	1	-1	0	3
0	3	0	-2	1	10

$$\begin{bmatrix} Z \\ X_2 \\ S_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 10 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} X_1 + \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} S_1$$

↑

S_1 is selected to enter the basis, but the minimum ratio test provides no "block" on the increase of S_1.



When a pivot column is selected (i.e., a variable is selected for increase) but there is no positive substitution rate on which to pivot, then the solution is unbounded.

If this is a "real-world" problem, this usually means that some error has been made in the formulation or in a previous pivot, since it is impossible to make unlimited profits!

Example

Maximize $X_1 + 2X_2 + 3X_3 + 4X_4$
subject to

$$\begin{cases} 2X_1 - X_2 + X_3 + 3X_4 \le 10 \\ X_2 + X_3 - X_4 \le 12 \\ X_1 + 2X_2 + 5X_3 + 2X_4 \le 20 \\ 4X_3 - X_4 \le 10 \end{cases}$$

$$X_j \ge 0,\ j=1,2,3,4$$

Initial Tableau

$-Z$	X_1	X_2	X_3	X_4	S_1	S_2	S_3	S_4	rhs
1	1	2	3	4	0	0	0	0	0
0	2	−1	1	3	1	0	0	0	10
0	0	1	1	−1	0	1	0	0	12
0	1	2	5	2	0	0	1	0	20
0	0	0	4	−1	0	0	0	1	10

(max)

-Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	S_4	rhs
1	¯1.67	3.33	1.67	0	¯1.33	0	0	0	¯13.3
0	0.667	¯0.333	0.333	1	0.333	0	0	0	3.33
0	0.667	0.667	1.33	0	0.333	1	0	0	15.3
0	¯0.333	2.67	4.33	0	¯0.667	0	1	0	13.3
0	0.667	¯0.333	4.33	0	0.333	0	0	1	13.3

(max)

-Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	S_4	rhs
1	¯1.54	2.31	0	0	¯1.08	0	¯0.385	0	¯18.5
0	0.692	¯0.538	0	1	0.385	0	¯0.0769	0	2.31
0	0.769	¯0.154	0	0	0.538	1	¯0.308	0	11.2
0	¯0.0769	0.615	1	0	¯0.154	0	0.231	0	3.08
0	1	¯3	0	0	1	0	¯1	1	0

(max)

$-Z$	X_1	X_2	X_3	X_4	S_1	S_2	S_3	S_4	rhs	
1	⁻1.25	0	⁻3.75	0	⁻0.5	0	⁻1.25	0	⁻30	(max)
0	0.625	0	0.875	1	0.25	0	0.125	0	5	
0	0.75	0	0.25	0	0.5	1	⁻0.25	0	12	
0	⁻0.125	1	1.63	0	⁻0.25	0	0.375	0	5	
0	0.625	0	4.88	0	0.25	0	0.125	1	15	

$-Z$	X_1	X_2	X_3	X_4	S_1	S_2	S_3	S_4	rhs	
1	⁻1.67	3.33	1.67	0	⁻1.33	0	0	0	⁻13.3	(max)
0	0.667	⁻0.333	0.333	1	0.333	0	0	0	3.33	
0	0.667	0.667	1.33	0	0.333	1	0	0	15.3	
0	⁻0.333	2.67	4.33	0	⁻0.667	0	1	0	13.3	
0	0.667	⁻0.333	4.33	0	0.333	0	0	1	13.3	

What if an error is made in selection of the pivot column?

Suppose that, after the first pivot produced the tableau above, we selected the X_1 column for the pivot.

-Z	X1	X2	X3	X4	S1	S2	S3	S4	rhs
1	-1.67	3.33	1.67	0	-1.33	0	0	0	-13.3
0	0.667	-0.333	0.333	1	0.333	0	0	0	3.33
0	0.667	0.667	1.33	0	0.333	1	0	0	15.3
0	-0.333	2.67	4.33	0	-0.667	0	1	0	13.3
0	0.667	-0.333	4.33	0	0.333	0	0	1	13.3

(max)

-Z	X1	X2	X3	X4	S1	S2	S3	S4	rhs
1	0	2.5	2.5	2.5	-0.5	0	0	0	-5
0	1	-0.5	0.5	1.5	0.5	0	0	0	5
0	0	1	1	-1	0	1	0	0	12
0	0	2.5	4.5	0.5	-0.5	0	1	0	15
0	0	0	4	-1	0	0	0	1	10

(max)

The objective is worse, not better!