Simplex Algorithm: an Introduction

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PAR, Inc. is a small manufacturer of golf equipment and supplies, including a

- STANDARD golf bag, and a

- DELUXE golf bag.
Each bag produced requires 4 operations, with the following processing times (hrs):

<table>
<thead>
<tr>
<th></th>
<th>Cut &amp; Dye</th>
<th>Sew</th>
<th>Finish</th>
<th>Inspect &amp; Pack</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>STANDARD</strong></td>
<td>$\frac{7}{10}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{1}{10}$</td>
</tr>
<tr>
<td><strong>DELUXE</strong></td>
<td>1</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>

After studying departmental workload projections, the plant manager estimates that the following time will be available for production of golf bags during the next quarter:

<table>
<thead>
<tr>
<th>Dept.</th>
<th>Man-hrs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cut-&amp;-Dye</td>
<td>630</td>
</tr>
<tr>
<td>Sewing</td>
<td>600</td>
</tr>
<tr>
<td>Finishing</td>
<td>708</td>
</tr>
<tr>
<td>Inspect-&amp;-Pack</td>
<td>135</td>
</tr>
</tbody>
</table>
PAR's distributor is convinced that everything which PAR makes can be easily sold, with a resulting profit of $10 per STANDARD bag and $9 per DELUXE bag.

PAR wishes to determine the number of each type bag which will maximize the profit.

**Definition of Variables**

\[ X_1 = \text{* STANDARD bags produced next qtr.} \]
\[ X_2 = \text{* DELUXE bags produced next qtr.} \]

**LP:** Maximize \[ 10X_1 + 9X_2 \]

subject to

\[ \frac{7}{10}X_1 + X_2 \leq 630 \]
\[ \frac{1}{2}X_1 + \frac{5}{6}X_2 \leq 600 \]
\[ X_1 + \frac{2}{3}X_2 \leq 708 \]
\[ \frac{1}{10}X_1 + \frac{1}{4}X_2 \leq 135 \]

\[ X_1 \geq 0 \quad X_2 \geq 0 \]
The shaded region satisfies all four inequality constraints!

Converting to "standard" LP model

Define "slack" variables

\[
\begin{align*}
S_1 &= \text{ unused hours in Cut-&-Dye Dept.} \\
S_2 &= \text{ unused hours in Sewing Dept.} \\
S_3 &= \text{ unused hours in Finishing Dept.} \\
S_4 &= \text{ unused hours in Inspect-&-Pack Dept.}
\end{align*}
\]

and

\[ Z = \text{ profit} \]
By the introduction of the "slack" variables, the inequalities (with the exception of the non-negativity restrictions) become equations:

Maximize
\[
\begin{align*}
10X_1 + 9X_2 & = Z \\
\frac{7}{10}X_1 + 1X_2 + S_1 & = 630 \\
\frac{1}{2}X_1 + \frac{5}{6}X_2 + S_2 & = 600 \\
1X_1 + \frac{2}{3}X_2 + S_3 & = 708 \\
\frac{1}{10}X_1 + \frac{1}{4}X_2 + S_4 & = 135 \\
\end{align*}
\]
subject to \(X_1 \geq 0\), \(X_2 \geq 0\)

Tableau

<table>
<thead>
<tr>
<th>-Z</th>
<th>X_1</th>
<th>X_2</th>
<th>S_1</th>
<th>S_2</th>
<th>S_3</th>
<th>S_4</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>\frac{7}{10}</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>630</td>
</tr>
<tr>
<td>0</td>
<td>\frac{1}{2}</td>
<td>\frac{5}{6}</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>600</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>\frac{2}{3}</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>708</td>
</tr>
<tr>
<td>0</td>
<td>\frac{1}{10}</td>
<td>\frac{1}{4}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>135</td>
</tr>
</tbody>
</table>
Notice that the system of equations represented by the tableau has essentially been "solved" for the variables $Z, S_1, S_2, S_3,$ and $S_4$ in terms of the variables $X_1$ and $X_2$:

\[
\begin{align*}
Z &= 0 + 10X_1 + 9X_2 \\
S_1 &= 630 - \frac{7}{10}X_1 - 1X_2 \\
S_2 &= 600 - \frac{1}{2}X_1 - \frac{5}{6}X_2 \\
S_3 &= 708 - 1X_1 - \frac{2}{3}X_2 \\
S_4 &= 135 - \frac{1}{10}X_1 - \frac{1}{4}X_2
\end{align*}
\]

If we assign arbitrary values to $X_1$ & $X_2$, we get "particular" solutions. For example, $X_1 = 100$ standard bags, $X_2 = 120$ deluxe bags.
If we let the "nonbasic" variables $X_1$ & $X_2$ be zero, then we obtain a "basic" solution:

$$
\begin{align*}
Z &= 0 + 10X_1 + 9X_2 \\
S_1 &= 630 - \frac{7}{10}X_1 - 1X_2 \\
S_2 &= 600 - \frac{1}{2}X_1 - \frac{5}{6}X_2 \\
S_3 &= 708 - 1X_1 - \frac{2}{3}X_2 \\
S_4 &= 135 - \frac{1}{10}X_1 - \frac{1}{4}X_2
\end{align*}
$$

(This basic solution is the plan to produce neither the STANDARD nor the DELUXE golf bags, resulting in all available production time being unused.)
This basic solution is one of the "corners" of the feasible region, which is a polyhedron.

Looking at the PROFIT equation,

\[ Z = 0 + 10X_1 + 9X_2 \]

we see that this basic solution is not optimal, since an increase in either \( X_1 \) or \( X_2 \) results in an increase in the profit \( Z \).

Let's arbitrarily select \( X_1 \) (i.e., production of the STANDARD golf bag) to be increased. Each unit of increase in \( X_1 \) results in a $10 increase in \( Z \) (profit).
As $X_1$ is increased, the values of the basic variables $S_1, S_2, S_3,$ and $S_4$ are also altered.  

\[
\begin{align*}
S_1 &= 630 - \frac{7}{10} X_1 - \ldots \\
S_2 &= 600 - \frac{1}{2} X_1 - \ldots \\
S_3 &= 708 - 1 X_1 - \ldots \\
S_4 &= 135 - \frac{1}{10} X_1 - \ldots
\end{align*}
\]

An increase of 1 unit of $X_1$ results in:
- $\frac{7}{10}$ unit decrease in $S_1$
- $\frac{1}{2}$ unit decrease in $S_2$
- 1 unit decrease in $S_3$
- $\frac{1}{10}$ unit decrease in $S_4$

How much may $X_1$ be increased?

A further increase in $X_1$ is "blocked" when one of the (currently) basic variables reaches its lower bound (zero). To continue increasing $X_1$ would cause a violation in the nonnegativity of the basic variable.
An increase of 1 unit of $X_1$ \[ \Rightarrow \begin{cases} \frac{3}{10} \text{ unit decrease in } S_1 \\ \frac{1}{2} \text{ unit decrease in } S_2 \\ 1 \text{ unit decrease in } S_3 \\ \frac{1}{10} \text{ unit decrease in } S_4 \end{cases} \]

Current values:

\[
\begin{align*}
S_1 &= 630 \\
S_2 &= 600 \\
S_3 &= 708 \\
S_4 &= 135
\end{align*}
\]

\[
\begin{align*}
S_1 &= 630 - \frac{3}{10} X_1 \\
S_2 &= 600 - \frac{1}{2} X_1 \\
S_3 &= 708 - 1 X_1 \\
S_4 &= 135 - \frac{1}{10} X_1
\end{align*}
\]

\[
\begin{align*}
S_1 &= 630 - \frac{3}{10} X_1 \\
S_2 &= 600 - \frac{1}{2} X_1 \\
S_3 &= 708 - 1 X_1 \\
S_4 &= 135 - \frac{1}{10} X_1
\end{align*}
\]

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\begin{align*}
S_1 &= 630 - \frac{3}{10} X_1 \\
S_2 &= 600 - \frac{1}{2} X_1 \\
S_3 &= 708 - 1 X_1 \\
S_4 &= 135 - \frac{1}{10} X_1
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
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\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]

\[
\begin{align*}
X_1 &\leq \frac{630}{\frac{3}{10}} \\
X_1 &\leq \frac{600}{\frac{1}{2}} \\
X_1 &\leq \frac{708}{1} \\
X_1 &\leq \frac{135}{\frac{1}{10}}
\end{align*}
\]
As we increase $X_1$ from zero, the first "block" occurs at
\[
\min \{900, 1200, 708, 1350\}
\]
\[
= 708, \text{ where } S_3 \text{ becomes zero.}
\]

We now wish to "re-solve" the system of equations so that $X_1$ is a basic variable and $S_3$ is nonbasic (and therefore zero).

<table>
<thead>
<tr>
<th>Current tableau</th>
<th>$-Z$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>10</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$\frac{7}{10}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>630</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{5}{6}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>600</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{2}{3}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>708</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>135</td>
</tr>
</tbody>
</table>

"Pivot" on the element in the column of the new basic variable and the blocking row.

\[\begin{align*}
\text{Subtract } 10 \times \text{ROW4 from ROW1} \\
\text{Subtract } (\frac{7}{10}) \times \text{ROW4 from ROW2} \\
\text{Subtract } (\frac{1}{2}) \times \text{ROW4 from ROW3} \\
\text{Subtract } (\frac{1}{10}) \times \text{ROW4 from ROW5}
\end{align*}\]
New tableau resulting from the pivot

<table>
<thead>
<tr>
<th></th>
<th>-Z</th>
<th>X₁</th>
<th>X₂</th>
<th>S₁</th>
<th>S₂</th>
<th>S₃</th>
<th>S₄</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>½</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-10</td>
<td>0</td>
<td>-7080</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>8/15</td>
<td>1</td>
<td>0</td>
<td>-7/10</td>
<td>0</td>
<td>134.4</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>½</td>
<td>0</td>
<td>1</td>
<td>-½</td>
<td>0</td>
<td>246</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>708</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>11/60</td>
<td>0</td>
<td>0</td>
<td>-1/10</td>
<td>1</td>
<td>64.2</td>
<td></td>
</tr>
</tbody>
</table>

Basic Variables

A pivot corresponds to a move along an edge from one corner to another!
This is another representation of the same "complete" solution of the system of equations. For example, if we let $X_2=120$ and $S_3=528$, we get the same "particular" solution which was mentioned earlier.

The *basic* solution corresponding to this choice of basic variables is *different*, however:

$X_2 = 0$ and $S_3 = 0$ yield $Z = 7080$ \# and

\[
\begin{align*}
S_1 &= 134.4 \text{ hrs.} \\
S_2 &= 246 \text{ hrs.} \\
X_1 &= 708 \text{ bags} \\
S_4 &= 64.2 \text{ hrs.}
\end{align*}
\]
Note that the current basic solution is still not optimal, however, since increasing $X_2$ will further increase the profit:

$$Z = 7080 + \frac{1}{3}X_2 - 10S_3$$

The coefficient of a variable in the equation for the profit, $Z$, is called the "relative profit".

The variable $X_2$ is the only nonbasic variable with a positive relative profit, so we will select it to be increased.

\[
\begin{align*}
S_1 &= 134.4 - \frac{8}{15}X_2 + ... \\
S_2 &= 246 - \frac{1}{2}X_2 + ... \\
X_1 &= 708 - \frac{2}{3}X_2 - ... \\
S_4 &= 64.2 - \frac{11}{60}X_2 + ...
\end{align*}
\]

As before, we will increase the nonbasic variable until one of the basic variables reaches its lower bound (zero), which "blocks" any further increase in $X_2$. 
Nonnegativity of the basic variables provides bounds on $X_2$:

\[
\begin{align*}
S_1 &= 134.4 - \frac{8}{15} X_2 \geq 0 \\
S_2 &= 246 - \frac{1}{2} X_2 \geq 0 \\
X_1 &= 708 - \frac{2}{3} X_2 \geq 0 \\
S_4 &= 64.2 - \frac{11}{60} X_2 \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
X_2 &\leq \frac{134.4}{\frac{8}{15}} = 252 \\
X_2 &\leq \frac{246}{\frac{1}{2}} = 492 \\
X_2 &\leq \frac{708}{\frac{2}{3}} = 1062 \\
X_2 &\leq \frac{64.2}{\frac{11}{60}} = 350.18 \\
\end{align*}
\]

As soon as $X_2$ reaches the smallest of these bounds (in this case 252), any further increase is blocked, since it would force a basic variable (in this case $S_1$) to become negative.

---

**Minimum Ratio Test**

The increase of a nonbasic variable is blocked when it reaches the minimum of the ratios of right-hand-sides to positive substitution rates in the constraint rows.

The variable which is basic in the row with the minimum ratio will be replaced by the increased variable.
The pivot row is the row with the minimum ratio of rhs to (positive) substitution rate!

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-Z$</td>
<td>1</td>
<td>0</td>
<td>$\frac{7}{3}$</td>
<td>0</td>
<td>0</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>$\frac{8}{15}$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{7}{10}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>$\frac{2}{3}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{60}$</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{10}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Result of the pivot:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-Z$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-\frac{35}{8}$</td>
<td>0</td>
<td>$-\frac{11}{16}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\frac{15}{8}$</td>
<td>0</td>
<td>$-\frac{21}{16}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\frac{15}{16}$</td>
<td>1</td>
<td>$\frac{5}{32}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-\frac{19}{8}$</td>
<td>0</td>
<td>$\frac{15}{8}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\frac{11}{32}$</td>
<td>0</td>
<td>$\frac{9}{64}$</td>
<td>1</td>
</tr>
</tbody>
</table>
A pivot corresponds to a move along an edge from one corner to an adjacent corner:

At this new basic solution, the nonbasic variables $S_1$ & $S_3$ are zero, i.e., the first and third constraints are "tight".

The basic solution corresponding to this choice of basis is to produce 540 STANDARD golf bags and 252 DELUXE golf bags, with 120 and 18 hours unused in the sewing and the inspect & pack depts., respectively.
Looking at the equation for PROFIT, we see that the "relative profits" of the nonbasic variables are both negative:

\[ Z = 7668 - \frac{35}{8} S_1 - \frac{111}{16} S_3 \]

This means that any positive values assigned to the variables \( S_1 \) and \( S_3 \) will result in a profit of less than $7668.

Therefore, the current basic solution must be optimal!