Sensitivity Analysis in Linear Programming

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To illustrate sensitivity analysis of an LP problem, we will use a "Product Mix" example:

PRODUCT MIX PROBLEM

- Five products can be manufactured: *A, B, C, D, & E*
- Each product requires time on each of three machines:

Product	MACHINE *1*	*2*	*3*
A	12	8	5
B	7	9	10
C	8	4	7
D	10	0	3
E	7	11	2

machine time requirement (minutes/lb.)

- 128 hours per week are available on each machine

- Any amounts which are produced may be sold at prices of $5, $4, $5, $4, & $4 per pound, respectively
- Variable labor costs are $4 per hour for machines 1 & 2, and $3 per hour for machine 3
- Material costs for products A and C are $2 per pound, and for products B, D, and E: $1 per pound

How much of each product should be manufactured per week, in order to maximize profits?

Definition of Decision Variables

A = quantity of product *A* to be produced (lb/week)
B = quantity of product *B* to be produced (lb/week)
C = quantity of product *C* to be produced (lb/week)
D = quantity of product *D* to be produced (lb/week)
E = quantity of product *E* to be produced (lb/week)

LINDO output

```
:LOOK ALL

MAX     1.417 A + 1.43 B + 1.85 C + 2.183 D + 1.7 E
SUBJECT TO
    2)  12 A + 7 B + 8 C + 10 D + 7 E <=  7680
    3)  8 A + 9 B + 4 C + 11 E        <=  7680
    4)  5 A + 10 B + 7 C + 3 D + 2 E  <=  7680
END
```

$$\frac{128\ \text{hrs}}{\text{week}} \times \frac{60\ \text{minutes}}{\text{hour}} = \frac{7680\ \text{minutes}}{\text{week}}$$

: GO

OBJECTIVE FUNCTION VALUE

1) 1817.59985

VARIABLE	VALUE	REDUCED COST
A	0.000000	1.379667
B	0.000000	0.248334
C	512.000000	0.000000
D	0.000000	0.075334
E	512.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	0.225833
3)	0.000000	0.010833
4)	3072.000000	0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
A	1.417000	1.379667	INFINITY
B	1.430000	0.248334	INFINITY
C	1.850000	0.092857	0.041091
D	2.183000	0.075334	INFINITY
E	1.700000	0.113001	0.081250

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	7680.000	2671.304199	2792.727539
3	7680.000	4388.571289	3840.000000
4	7680.000	INFINITY	3072.000000

REDUCED COST *(LINDO's definition)*

> The reduced cost of a NONBASIC variable is the rate of "deterioration" of the objective function as the variable is forced to increase by a small amount.

Because of the optimality of the solution, the quantity reported in the LINDO output should be nonnegative!

In a MAXIMIZATION problem, the "reduced cost" gives the rate of ***decrease*** in the objective, while in a MINIMIZATION problem, it gives the rate of ***increase*** in the objective!

(The reduced cost of a BASIC variable is, of course, zero!)

Example: Suppose that, in the product mix example, it is necessary to produce ten pounds of product A for use as a sample for customers. What will be the cost (i.e., opportunity cost, or loss of profit) in doing this?

(Note that in the optimal solution, A=0, i.e., variable A is nonbasic.)

VARIABLE	VALUE	REDUCED COST	
A	0.000000	1.379667	☜
B	0.000000	0.248334	
C	512.000000	0.000000	← *basic*
D	0.000000	0.075334	
E	512.000000	0.000000	← *basic*

The profit will be reduced by 10 pounds x $1.379667/pound

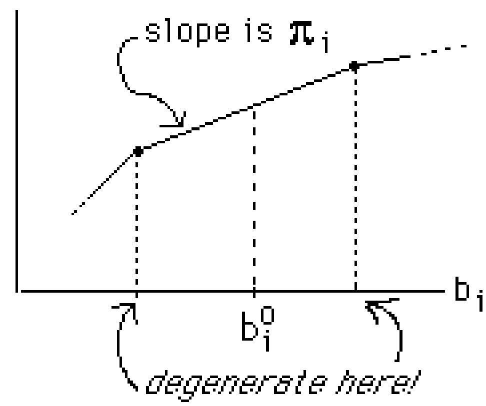
Dual Variable

(also known as "simplex multiplier")

> The dual variable π_i for row i may be considered as the rate of change of the optimal objective value Z* with respect to the right-hand-side b_i

i.e., $\pi_i = \dfrac{\partial Z^*}{\partial b_i}$

This is true only for cases where the current solution is non-degenerate; otherwise Z is not differentiable.*

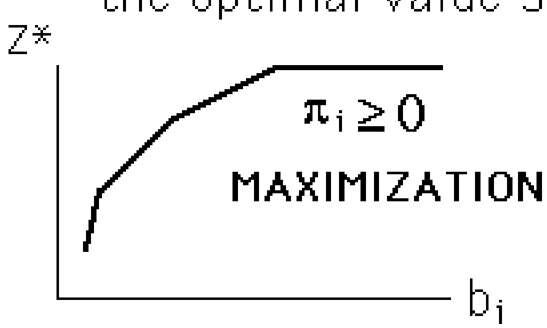


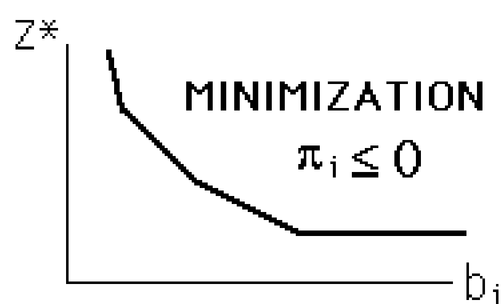
Dual Variable

$\pi_i = \dfrac{\partial Z^*}{\partial b_i}$

If a constraint is of type $\sum_{j=1}^{n} a_{ij}X_j \leq b_i$

then as the right-hand-side b_i ***increases***, the constraint becomes ***less*** restrictive, and so the optimal value should ***improve***.



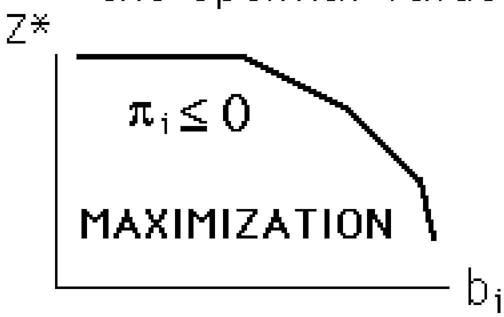


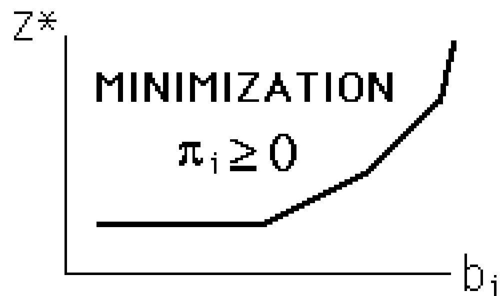
Dual Variable

$$\pi_i = \frac{\partial Z^*}{\partial b_i}$$

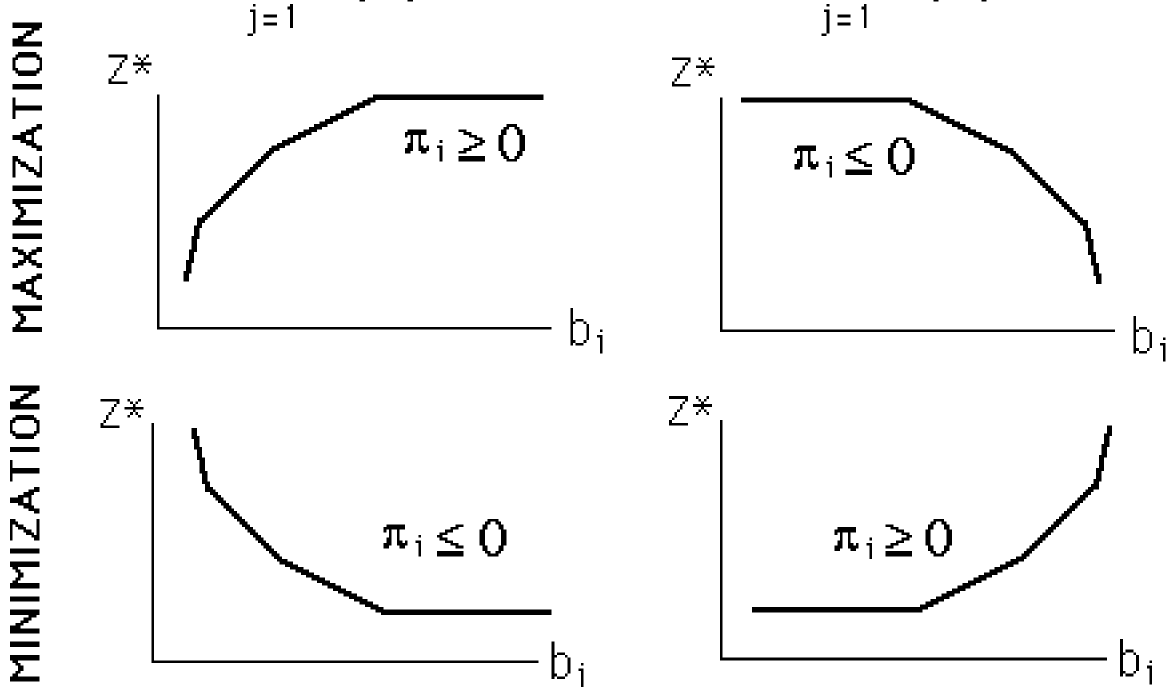
If a constraint is of type $\sum_{j=1}^{n} a_{ij}X_j \geq b_i$

then as the right-hand-side b_i ***increases***, the constraint becomes ***more*** restrictive, and so the optimal value should ***deteriorate***.





$$\sum_{j=1}^{n} a_{ij}X_j \leq b_i \qquad\qquad \sum_{j=1}^{n} a_{ij}X_j \geq b_i$$



Sign of Dual Variable

$$\pi_i = \frac{\partial Z^*}{\partial b_i}$$

Type of Constraint (columns) / **Type of Objective** (rows)

Type of Objective	$\leq$	=	$\geq$
MIN	≤ 0	?	≥ 0
MAX	≥ 0	?	≤ 0

Dual Price

LINDO reports, instead of the dual variables, a quantity called "dual price" defined as:

> *Dual Price of a constraint: the rate at which the objective value will "improve" as the right-hand-side of the constraint is increased by a small amount*

"improve" = { increase in a MAX problem; decrease in a MIN problem }

MAX problem:
"dual price" = "dual variable"

MIN problem:
"dual price" = −dual variable

In our product mix example, the optimal solution report provided by LINDO includes the following dual prices:

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	0.225833
3)	0.000000	0.010833
4)	3072.000000	0.000000

This indicates that (since the current optimal solution is not degenerate, i.e., there are three positive basic variables: C, E, and SLK 4, and three constraints.) the profit will increase $0.225833 per additional minute of available time on machine #1.

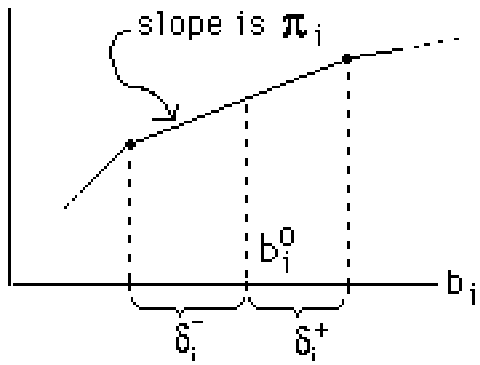
Note that if a constraint is not "tight", i.e., if the slack (or surplus) is positive in that constraint, the dual price must be zero:

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	0.225833
3)	0.000000	0.010833
4)	3072.000000	0.000000 ←

dual price is 0 if constraint is not tight!

(This is a consequence of the complementary slackness theorem, but is understandable in view of the economic interpretation of the dual prices: if a resource is not completely used at the optimum, then additional quantities of that resource will not result in increased profit or decreased costs.)

If the right-hand-side b_i is in the interval $[\, b_i^0 - \delta_i^-,\ b_i^0 + \delta_i^+ \,]$,
where b_i^0 = current value of RHS in row i
δ_i^- = allowable decrease
δ_i^+ = allowable increase
then the current basis (B) remains optimal, and $\pi = c_B (A^B)^{-1}$



The LINDO output includes the following RHS ranges:

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	7680.000	2671.304199	2792.727539
3	7680.000	4388.571289	3840.000000
4	7680.000	INFINITY	3072.000000

Substitution Rates

The coefficients in the optimal tableau may be interpreted as "substitution rates" in the following sense:

Suppose that the coefficient of a nonbasic variable X_j in row i is α_{ij}.

Then as X_j **in**creases, the basic variable in row i **de**creases at the rate α_{ij} (per unit of X_j).
(If $\alpha_{ij} < 0$, then this means an **in**crease in the basic variable!)

That is, one unit of X_j "substitutes for" α_{ij} units of the basic variable in the optimal solution.

The optimal tableau for the Product-Mix example is, according to LINDO:

: TABLEAU

ROW	(BASIS)	A	B	C	D
1	ART	1.380	0.248	0.000	0.075
2	C	1.267	0.233	1.000	1.833
3	E	0.267	0.733	0.000	-0.667
4	SLK 4	-4.400	6.900	0.000	-8.500

ROW	E	SLK 2	SLK 3	SLK 4	
1	0.000	0.226	0.011	0.000	1817.600
2	-0.000	0.183	-0.117	0.000	512.000
3	1.000	-0.067	0.133	0.000	512.000
4	0.000	-1.150	0.550	1.000	3072.000

The ***complete solution*** can be written, according to the output of the TABLEAU command in LINDO, as

$$\begin{bmatrix} \text{PROFIT} \\ C \\ E \\ \text{SLK 4} \end{bmatrix} = \begin{bmatrix} 1817.6 \\ 512.0 \\ 512.0 \\ 3072.0 \end{bmatrix} - \begin{bmatrix} 1.380 \\ 1.267 \\ 0.267 \\ -4.400 \end{bmatrix} A - \begin{bmatrix} 0.248 \\ 0.233 \\ 0.733 \\ 6.900 \end{bmatrix} B - \begin{bmatrix} 0.075 \\ 1.833 \\ -0.667 \\ -8.500 \end{bmatrix} D - \begin{bmatrix} 0.226 \\ 0.183 \\ -0.067 \\ -1.150 \end{bmatrix} S_2 - \begin{bmatrix} 0.011 \\ -0.117 \\ 0.133 \\ 0.550 \end{bmatrix} S_3$$

If we were to produce ten units of product A (even though it is not optimal to do so!), what will be the effect on:

the profit?
the amount of C to be produced?
the amount of E to be produced?
the amount of unused time on machine #3?

If the nonbasic variable A increases by 10 units,

$$\begin{bmatrix} \text{PROFIT} \\ C \\ E \\ \text{SLK 4} \end{bmatrix} = \begin{bmatrix} 1817.6 \\ 512.0 \\ 512.0 \\ 3072.0 \end{bmatrix} - \begin{bmatrix} 1.380 \\ 1.267 \\ 0.267 \\ -4.400 \end{bmatrix} A - \ldots$$

Profit will decrease by 10x 1.380 ($)
Production of C will decrease by 10x 1.267
Production of E will decrease by 10x 0.267
Slack time on machine#3 will increase by 10x 4.400 (minutes)

If four hours of overtime are available on any machine of your choice, at a cost of $10/hour (i.e., $0.1667/minute),

- *should it be used?*
- *on which machine or machines should it be used?*
- *how would the production plan change?*

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	0.225833 ← *greater than the cost of overtime (0.1667/minute)*
3)	0.000000	0.010833
4)	3072.000000	0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	7680.000	2671.304199	2792.727539
3	7680.000	4388.571289	3840.000000
4	7680.000	INFINITY	3072.000000

more than four hours (240 min.)

Clearly, increasing the RHS of row 2 (i.e., machine #1 available time) by 240 minutes will increase the profit more than enough to offset the overtime cost.

To determine the effect on the production plan of increasing machine #1 usage by 240 minutes, we must decrease the slack variable (unused time on machine #1) from zero to ***negative*** *240 minutes.*

2)	12 A + 7 B + 8 C + 10 D + 7 E	+ SLK2	= 7680
Current solution:	7680	0	= 7680
Revised solution:	7920	-240	= 7680

(The solution we obtain will, of course, have negative slack in the machine availability constraint for machine #1, and will be infeasible in the original statement of the problem!)

The "substitution rates" for SLK 2, i.e., slack variable S_2*, are found in the TABLEAU:*

ROW	(BASIS)	A	B	C	D
1	ART	1.380	0.248	0.000	0.075
2	C	1.267	0.233	1.000	1.833
3	E	0.267	0.733	0.000	-0.667
4	SLK 4	-4.400	6.900	0.000	-8.500

ROW	E	SLK 2	SLK 3	SLK 4	
1	0.000	0.226	0.011	0.000	1817.60
2	-0.000	0.183	-0.117	0.000	512.00
3	1.000	-0.067	0.133	0.000	512.00
4	0.000	-1.150	0.550	1.000	3072.00

That is,

$$\begin{bmatrix} \text{PROFIT} \\ \text{C} \\ \text{E} \\ \text{SLK 4} \end{bmatrix} = \begin{bmatrix} 1817.6 \\ 512.0 \\ 512.0 \\ 3072.0 \end{bmatrix} - \begin{bmatrix} 0.226 \\ 0.183 \\ -0.067 \\ -1.150 \end{bmatrix} S_2$$

$$\begin{bmatrix} PROFIT \\ C \\ E \\ SLK\ 4 \end{bmatrix} = \begin{bmatrix} 1817.6 \\ 512.0 \\ 512.0 \\ 3072.0 \end{bmatrix} - \begin{bmatrix} 0.226 \\ 0.183 \\ -0.067 \\ -1.150 \end{bmatrix}(-240)$$

$$\begin{aligned} PROFIT &= 1817.6 + 240(0.226) = 1817.6 + 52.24 = 1869.84 \\ C &= 512 + 240(0.183) = 512 + 43.92 = 555.92 \\ E &= 512 - 240(0.067) = 512 - 18 = 494 \\ SLK\ 4 &= 3072 - 240(1.15) = 3072 - 276 = 2796 \end{aligned}$$

From the profit above, we must subtract the cost of four hours' overtime at $10/hour, i.e., the net profit will be $1829.84

(Note that, with the use of overtime, the output of product E ***de****creases, while, as one might expect, that of C increases!)*

We can now see how the "ALLOWABLE INCREASE" in the RHS Range output is computed:

$$\begin{bmatrix} PROFIT \\ C \\ E \\ SLK\ 4 \end{bmatrix} = \begin{bmatrix} 1817.6 \\ 512.0 \\ 512.0 \\ 3072.0 \end{bmatrix} - \begin{bmatrix} 0.226 \\ 0.183 \\ -0.067 \\ -1.150 \end{bmatrix} S_2$$

In order for the current basis to remain unchanged, we must have

$$0 \le C = 512 - 0.183S_2 \Rightarrow S_2 \le \frac{512}{0.183} = 2792.727$$

$$0 \le E = 512 + 0.067S_2 \Rightarrow S_2 \ge \frac{-512}{0.067} = -7679.961$$

$$0 \le SLK\ 4 = 3072 + 1.150S_2 \Rightarrow S_2 \ge \frac{-3072}{1.150} = -2671.304 \leftarrow \text{Greatest Lower Bound}$$

ALLOWABLE INCREASE	ALLOWABLE DECREASE
2671.304	2792.727

(The allowable decrease in the slack variable = allowable increase in the RHS!)

In general, in the RHS of constraint #j is changed by an amount δ, then the vector of basic variables $X_B(\delta)$ is changed from its current value of $\hat{X}_B$ to

$$X_B(\delta) = (A^B)^{-1}[b+\Delta] \quad \text{where} \quad \Delta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \delta \\ \vdots \\ 0 \\ 0 \end{bmatrix} \leftarrow \textit{row } j$$

$$= (A^B)^{-1} b + (A^B)^{-1}\Delta = \hat{X}_B + \left(\begin{smallmatrix}\textit{column } j \textit{ of} \\ \textit{basis inverse}\end{smallmatrix}\right)\delta$$

Let β_{ij} denote the element in row #i, column #j of the basis inverse matrix. Then the basic variable in row #i, $X_{B(i)}$, is

$$X_{B(i)}(\delta) = \hat{X}_{B(i)} + \beta_{ij}\delta$$

In order that the basis B remains optimal, it must be feasible, and so the basic variables must be nonnegative:

$$0 \le X_{B(i)}(\delta) = \hat{X}_{B(i)} + \beta_{ij}\delta$$

$$\Rightarrow \quad \beta_{ij}\delta \ge -\hat{X}_{B(i)} \Rightarrow \begin{cases} \delta \ge \dfrac{-\hat{X}_{B(i)}}{\beta_{ij}} & \textit{if} \quad \beta_{ij} > 0 \\[2ex] \delta \le \dfrac{-\hat{X}_{B(i)}}{\beta_{ij}} & \textit{if} \quad \beta_{ij} < 0 \end{cases}$$

$$\Rightarrow \quad \underset{\beta_{ij} > 0}{\textit{Maximum}} \left\{\frac{-\hat{X}_{B(i)}}{\beta_{ij}}\right\} \le \delta \le \underset{\beta_{ij} < 0}{\textit{Minimum}} \left\{\frac{-\hat{X}_{B(i)}}{\beta_{ij}}\right\}$$

ALLOWABLE DECREASE *ALLOWABLE INCREASE*

In addition to RIGHT-HAND-SIDE RANGES, LINDO will also provide us with OBJECTIVE COEFFICIENT RANGES. For the Product-Mix example, these are:

RANGES IN WHICH THE BASIS IS UNCHANGED

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
A	1.417000	1.379667	INFINITY
B	1.430000	0.248334	INFINITY
C	1.850000	0.092857	0.041091
D	2.183000	0.075334	INFINITY
E	1.700000	0.113001	0.081250

If an objective coefficient remains within the range given by the ALLOWABLE INCREASE & ALLOWABLE DECREASE, then the current basis (& the current basic solution) remains optimal.

Warning: This assumes that a SINGLE coefficient is changed, while the others are fixed.

How are these objective coefficient ranges computed? Suppose that the coefficient of variable k changes by an amount δ (which may be positive or negative):

$$c'_k = c_k + \delta$$

There are two cases to consider:

CASE ONE: The variable X_k is NONBASIC

CASE TWO: The variable X_k is BASIC

Since the change being considered has no effect on the feasibility of the current basic solution, we need to consider only how the change effects the optimality conditions, i.e.,

reduced cost ≥ 0 (if minimizing)

relative profit ≥ 0 (if maximizing)

CASE ONE: The variable X_k is NONBASIC

This is the simpler case, since the change in c_k effects on the optimality condition for the nonbasic variable X_k:
The new reduced cost is

$$c_k' - \pi A^k = (c_k + \delta) - \pi A^k = (c_k - \pi A^k) + \delta \quad = \text{old reduced cost} + \delta$$

and this remains nonnegative if $\delta \geq$ – old relative profit

or, if we are maximizing, the relative profit remains nonpositive if

$$\delta \leq - \text{old relative profit}$$

For the Product Mix problem, variables A, B, and D are nonbasic:

VARIABLE	VALUE	REDUCED COST
A	0.000000	1.379667
B	0.000000	0.248334
C	512.000000	0.000000
D	0.000000	0.075334
E	512.000000	0.000000

In a MAX problem, the relative profits are ≤ 0 at the optimum. The relative profits are the negatives of the numbers shown.

RANGES IN WHICH THE BASIS IS UNCHANGED

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
A	1.417000	1.379667	INFINITY
B	1.430000	0.248334	INFINITY
C	1.850000	0.092857	0.041091
D	2.183000	0.075334	INFINITY
E	1.700000	0.113001	0.081250

There is no bound on the decrease

CASE TWO: *The variable X_k is BASIC*

This is the more difficult case to analyze, since a change in c_k will change the simplex multipliers, and therefore will effect the reduced costs of all of the variables!

If $c'_k = c_k + \delta$ and variable X_k is basic in row #i, then the new simplex multiplier vector is

$$\pi' = c'_B (A^B)^{-1} = (c_B + \Delta)(A^B)^{-1} \quad \text{where } \Delta = [0\ 0\ \ldots\ \delta\ \ldots\ 0\ 0]$$

(δ is component # i)

$$= c_B (A^B)^{-1} + \Delta (A^B)^{-1} = \pi + \Delta (A^B)^{-1}$$

$$= \pi + \left(\text{row \# i of the basis inverse}\right)\delta$$

The new reduced cost of variable X_j ($j \neq k$) will be

$$c_j - \pi' A^j = c_j - [\pi + \Delta (A^B)^{-1}] A^j = [c_j - \pi A^j] - \Delta (A^B)^{-1} A^j$$

($c_j - \pi A^j$ is the old reduced cost)

$$= \hat{c}_j - \Delta \hat{A}^j$$

($\hat{A}^j$ is the column of substitution rates)

$$= \hat{c}_j - \alpha_{ij}\delta$$

This reduced cost is nonnegative if

$$0 \leq \hat{c}_j - \alpha_{ij}\delta \Rightarrow \alpha_{ij}\delta \leq \hat{c}_j \Rightarrow \begin{cases} \delta \leq \dfrac{\hat{c}_j}{\alpha_{ij}} & \text{if } \alpha_{ij} > 0 \\[2ex] \delta \geq \dfrac{\hat{c}_j}{\alpha_{ij}} & \text{if } \alpha_{ij} < 0 \end{cases}$$

$$\Rightarrow \quad \underset{\alpha_{ij} < 0}{\textbf{Maximum}} \left\{ \frac{\hat{c}_j}{\alpha_{ij}} \right\} \leq \delta \leq \underset{\alpha_{ij} > 0}{\textbf{Minimum}} \left\{ \frac{\hat{c}_j}{\alpha_{ij}} \right\}$$

If we are maximizing, the relative profit must remain ≤ 0, i.e.,

$$0 \geq \hat{c}_j - \alpha_{ij}\delta \Rightarrow \alpha_{ij}\delta \geq \hat{c}_j$$

$$\Rightarrow \begin{cases} \delta \geq \dfrac{\hat{c}_j}{\alpha_{ij}} & \text{if } \alpha_{ij} > 0 \\ \delta \leq \dfrac{\hat{c}_j}{\alpha_{ij}} & \text{if } \alpha_{ij} < 0 \end{cases}$$

$$\underset{\alpha_{ij} > 0}{\textbf{\textit{Maximum}}} \left\{ \frac{\hat{c}_j}{\alpha_{ij}} \right\} \leq \delta \leq \underset{\alpha_{ij} < 0}{\textbf{\textit{Minimum}}} \left\{ \frac{\hat{c}_j}{\alpha_{ij}} \right\}$$

Consider again the Product-Mix problem. Variable C is basic in the second row.

The optimal tableau, according to LINDO, is

ROW	(BASIS)	A	B	C	D
1	ART	1.380	0.248	0.000	0.075
2	C	1.267	0.233	1.000	1.833
3	E	0.267	0.733	0.000	-0.667
4	SLK 4	-4.400	6.900	0.000	-8.500

ROW	E	SLK 2	SLK 3	SLK 4	
1	0.000	0.226	0.011	0.000	1817.600
2	-0.000	0.183	-0.117	0.000	512.000
3	1.000	-0.067	0.133	0.000	512.000
4	0.000	-1.150	0.550	1.000	3072.000

Since this is a MAXimization problem, the relative profits must be nonpositive at the optimum; therefore, it appears that LINDO has reversed the signs in the objective row.

Computation of the objective coefficient range for basic variable C:

ALLOWABLE INCREASE

$$\delta \leq \text{Minimum}\left\{ \frac{-0.011}{-0.117} \right\} = 0.092857$$

ALLOWABLE DECREASE

$$\delta \geq \text{Maximum}\left\{ \frac{-1.380}{1.267}, \frac{-0.248}{0.233}, \frac{-0.075}{1.833}, \frac{-0.226}{0.183} \right\}$$

$$\delta \geq \text{Maximum}\{-1.089187, -1.064378, -.041091, -1.234973\}$$

$$\delta \geq -0.041091$$

Therefore, if the profit coefficient of the variable C increases by an amount no greater than $0.092857/pound

or decreases by an amount no greater than $0.041091/pound, the current solution will remain optimal.