Random Number Generation

author

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- Generating Uniformly-Distributed Numbers
- Inverse Transformation Method
- Rejection Method

Generating Uniformly-Distributed Numbers

Beginning with a "seed" X_0, a sequence of random numbers $X_1, X_2, X_3, \ldots$ is generated by some operation

$$X_{i+1} = G(X_i), \quad i=1,2,3,\ldots$$

Since the sequence is determined by the operator G and the initial "seed", the numbers are not, in fact, random, but if uniformly distributed in the interval [0,1] they will be called "pseudo-random".

Example *"Midsquare Technique"*

Determine X_{i+1} as follows:

Compute X_i^2, discard the final 2 digits, and take the last 4 digits of the remaining number:

i	X_i	X_i^2
0	1912	03 6557 44
1	6557	42 9942 49
2	9942	98 8433 64
3	8433	71 1154 89
4	1154	01 3317 16

Example *Congruential Method*

Choose $A>0$.

Determine X_{i+1} by the operation:

$$X_{i+1} = [AX_i] \text{ Modulo } M$$

i.e., multiply X_i by A and divide the result by M.
Keep the remainder of the division, and call it X_{i+1}.

Example $X_{i+1} = [AX_i] \text{ Modulo } M$

Select
$X_0 = 5$,
$A=5$,
$M=17$

i	X_i	AX_i	$\lfloor AX_i \div M \rfloor$	remainder
0	5	25	1	8
1	8	40	2	6
2	6	30	1	13
3	13	65	3	14
4	14	70	4	2
5	2	10	0	10
⋮	⋮	⋮	⋮	⋮

Usually, $M = 2^B - 1$, where B=# bits/word for the computer used

E.g., for IBM system 360, the choice was

$$M = 2^{32} - 1$$
$$A = X_0 = 3^{19} = 65539$$

(with these values, the sequence repeats after 2^{39} numbers!)

Uniformly-distributed Random Number Table

3821	4876	3071	5268	8684	0169	1746	6658	8605	9638
0218	3519	0707	3695	6478	3977	2017	3644	7993	5547
5105	8147	7365	2901	7228	2307	7241	4225	6078	9344
4549	1468	4395	3808	9446	5954	6851	2930	9217	5668
6758	7233	0503	0981	5955	4881	5916	3197	8532	9810
8431	5742	0744	3115	4411	5132	2175	8044	5668	3463
5072	1129	0723	1390	0722	6669	8144	0434	3014	9675
1797	8050	3603	9301	2162	8267	6733	5878	9918	3984
5280	5063	6663	6449	6400	0863	2414	4309	0851	3393
7223	4603	1542	9279	7217	2279	4575	5332	0000	6645

Inverse Transformation Method

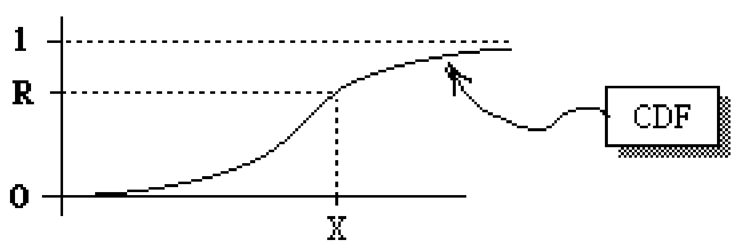
$F(x)$ = cumulative distribution function (CDF) of probability distribution to be simulated $= P\{X \le x\}$

R = random variable ***uniformly*** distributed in the interval $[0,1]$

Randomly generate R (uniformly dist'd in $[0,1]$)

Find X such that $F(X) = R$

i.e., $X = F^{-1}(R)$ ← *inverse of CDF*



X generated in this way has the desired distribution

Example

Exponential Distribution

$$F(x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

$$F(x) = 1 - e^{-\lambda x} = R$$

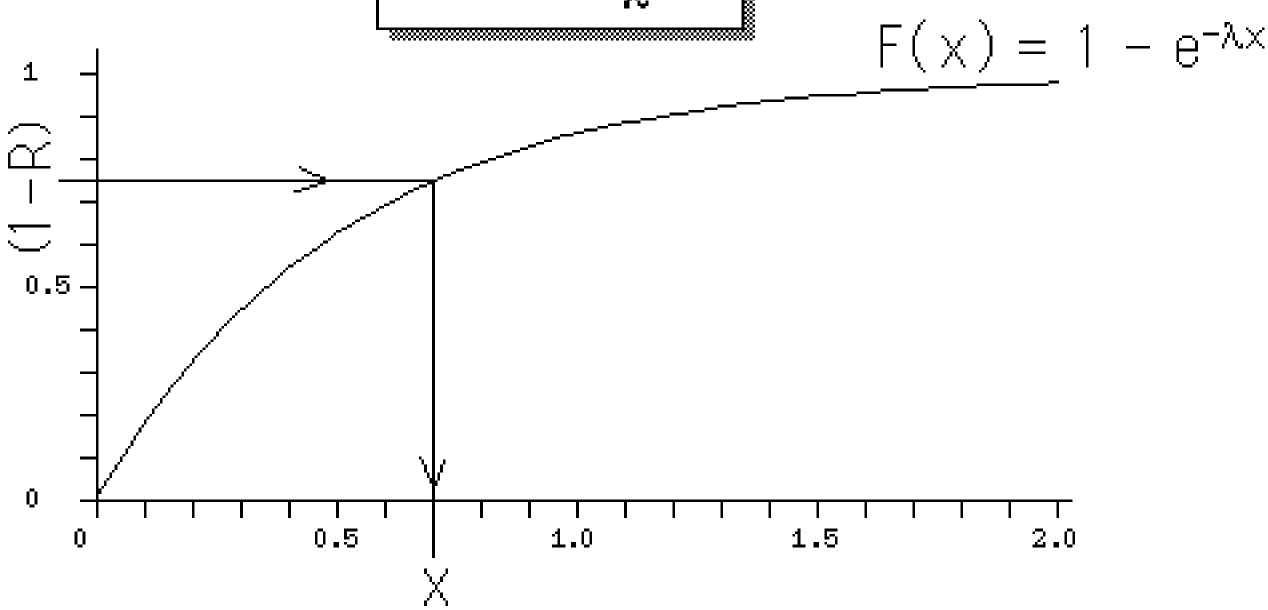
$$e^{-\lambda x} = 1 - R = \overline{R}$$

$$-\lambda x = \ln \overline{R}$$

Both R & 1-R are uniformly distributed in [0,1]

$$x = -\frac{\ln \overline{R}}{\lambda}$$

$$X = -\frac{\ln(1-R)}{\lambda}$$



Suppose that we wish to simulate a Poisson process with $\lambda = 2/\text{hr}$. For this purpose, we need to randomly generate the arrival times

$$T_1, T_2, T_3, \ldots$$

The time between arrivals will have the exponential distribution with parameter $\lambda = 2/\text{hr}$. We will randomly generate values having this distribution.

$\bar{R}$
3821
0218
5105
4549
6758
8431
5072
1797
5280
7223

Inverse Transformation Method for Exponential Distribution

Let's use the first column of the uniformly distributed random number table earlier in this stack.

$\bar{R} = 0.3821 \Longrightarrow$

$$X = -\frac{\ln 0.3821}{2} = -\frac{(-0.9621)}{2} = 0.4810$$

$$\boxed{X = -\frac{\ln \bar{R}}{\lambda}}$$

Generating the first 3 random values

$\overline{R}$
3821
0218
5105
4549
6758
8431
5072
1797
5280
7223

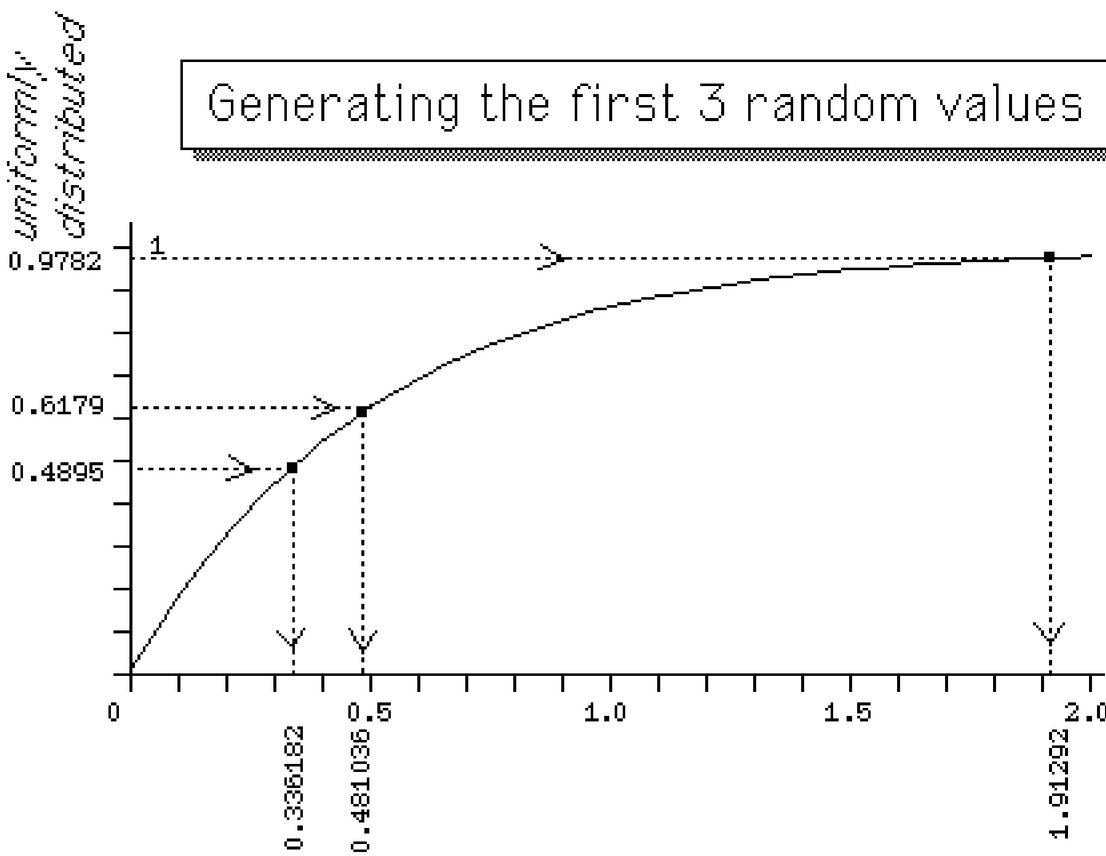
$$X = -\frac{\ln 0.3821}{2} = -\frac{(-0.9621)}{2} = 0.4810$$

$$X = -\frac{\ln 0.0218}{2} = -\frac{(-3.8258)}{2} = 1.9129$$

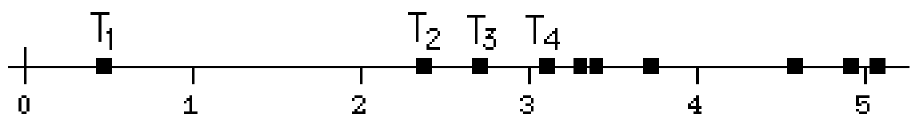
$$X = -\frac{\ln 0.5105}{2} = -\frac{(-0.6724)}{2} = 0.3362$$

$$X = -\frac{\ln \overline{R}}{\lambda}$$

Generating the first 3 random values



$\overline{R}_i$	R_i	X_i	$T_i = \sum_{k=1}^{i} X_k$
0.3821	0.6179	0.481036	0.481036
0.0218	0.9782	1.91292	2.39396
0.5105	0.4895	0.336182	2.73014
0.4549	0.5451	0.393839	3.12398
0.6758	0.3242	0.195929	3.31991
0.8431	0.1569	0.0853349	3.40524
0.5072	0.4928	0.339425	3.74467
0.1797	0.8203	0.858233	4.6029
0.528	0.472	0.319329	4.92223
0.7223	0.2777	0.162657	5.08489

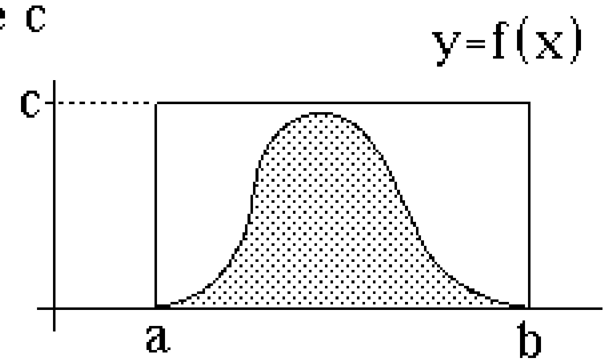


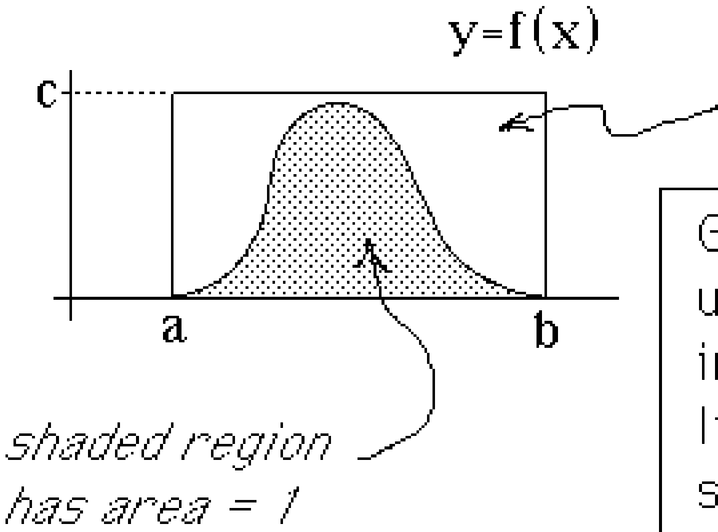
Simulated Arrival Times

Rejection Method

Generates sample values for any random variable that

- assumes values only within a finite interval [a,b]
- has a density function that is bounded by a finite value c





Generate a point uniformly distributed in the rectangle.
If it lies within the shaded region, then accept x-coordinate as the generated number. Repeat as necessary.

Algorithm

1) Generate 2 random numbers $\mathbf{R}_1$ & $\mathbf{R}_2$ uniformly distributed in [0,1]

2) Let $\mathbf{X} = (b-a)\mathbf{R}_1 + a$ and $\mathbf{Y} = c\mathbf{R}_2$ to get a point $(\mathbf{X},\mathbf{Y})$ uniformly distributed in the rectangle

3) Accept $\mathbf{X}$ if $\mathbf{Y} \le f(\mathbf{X})$, i.e., the point lies in the shaded region under the graph of y=f(x). Otherwise, reject $\mathbf{X}$ and return to step 1.

Example Beta Distribution

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1$$

$$\text{Mode} = \frac{\alpha-1}{\alpha+\beta-2}$$

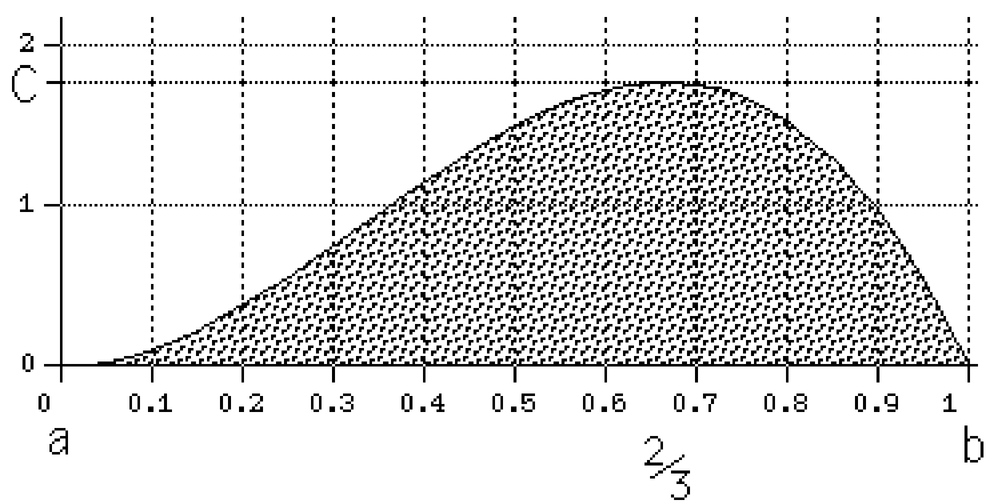
$$\mu = \frac{\alpha}{\alpha+\beta}$$

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Suppose $\alpha = 3$, $\beta = 2$.

a=0, b=1

$C = f(2/3) = 1.77777778$



Select the second column from the table of uniformly-distributed random numbers:

4876
3519
8147
1468
7233
5742
1129
8050
5063
4603

The first 2 uniformly-distributed random numbers are

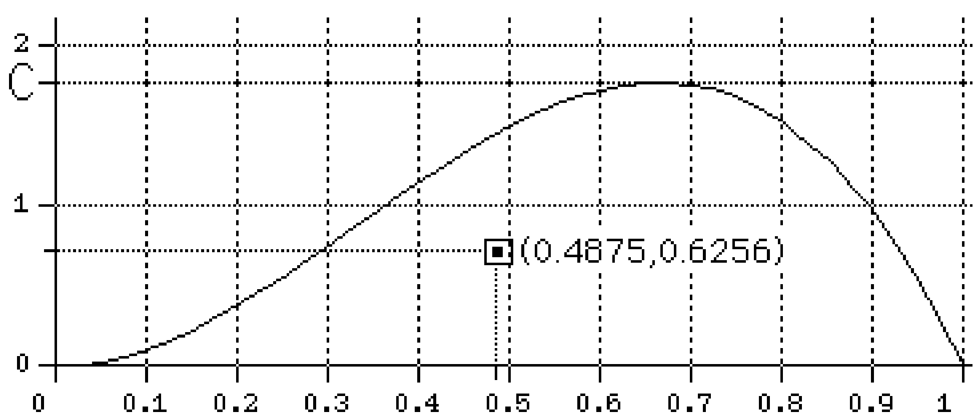
$R_1 = 0.4875,$

$R_2 = 0.3519$

$X = R_1 = 0.4875,$

$Y = R_2 C = 0.3519 \times 1.7777778 = 0.6256$

$f(X) = 1.4619 > Y$ ACCEPT!



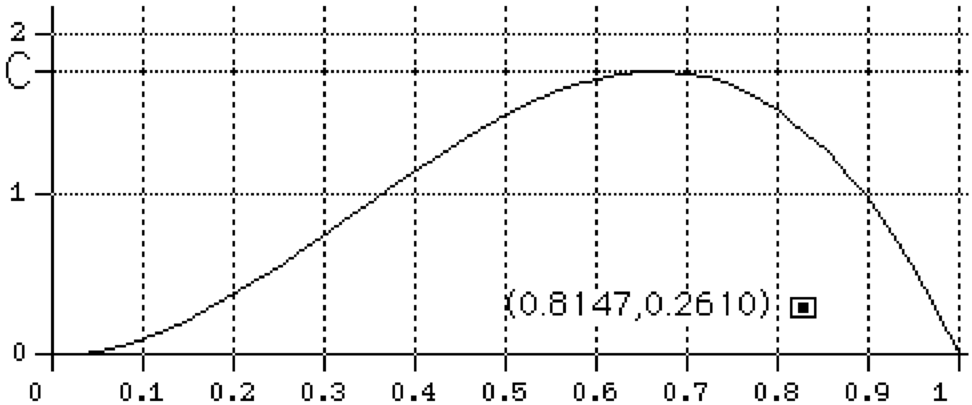
the point (x_i, y_i) is under the density curve!

The next 2 uniformly-distributed random numbers are

$R_1 = 0.8147$, $R_2 = 0.1468$

$X = 0.8147$, $Y = R_2 C = 0.260978$

$f(X) = 1.47588 > Y$ **ACCEPT!**



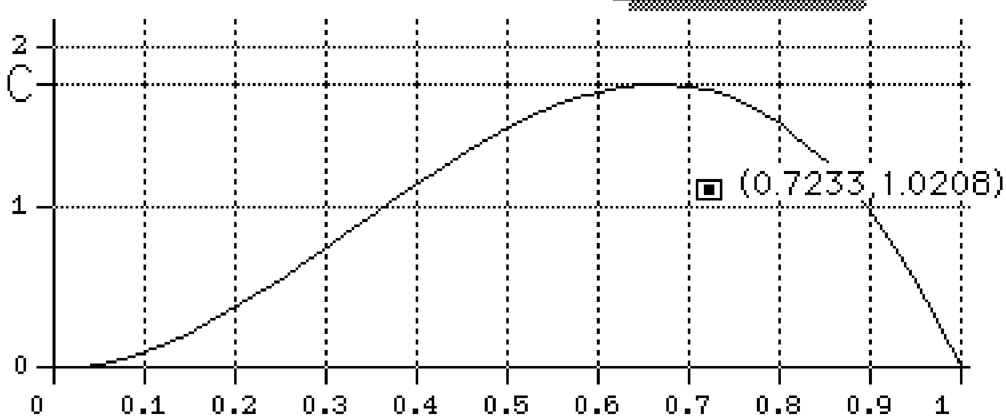
again, the point (x,y) is under the density curve!

The next 2 uniformly-distributed random numbers are

$R_1 = 0.7233$, $R_2 = 0.5742$

$X = 0.7233$, $Y = R_2 C = 1.0208$

$f(X) = 1.73711 > Y$ **ACCEPT!**



again, the point (x,y) is under the density curve!

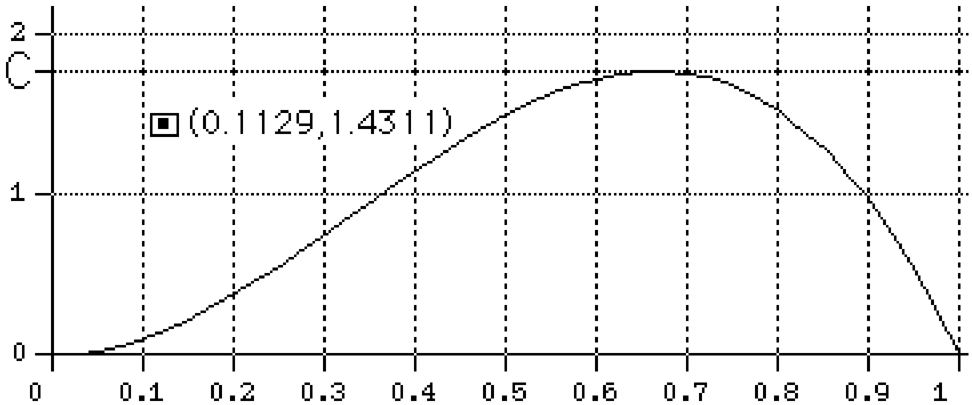
The next 2 uniformly-distributed random numbers are

$R_1 = 0.1129$, $R_2 = 0.8050$

$X = 0.1129$, $Y = R_2C = 1.4311$

$f(X) = 0.13569 < Y$ **REJECT!**

the point (x,y) is NOT under the curve, and is rejected!



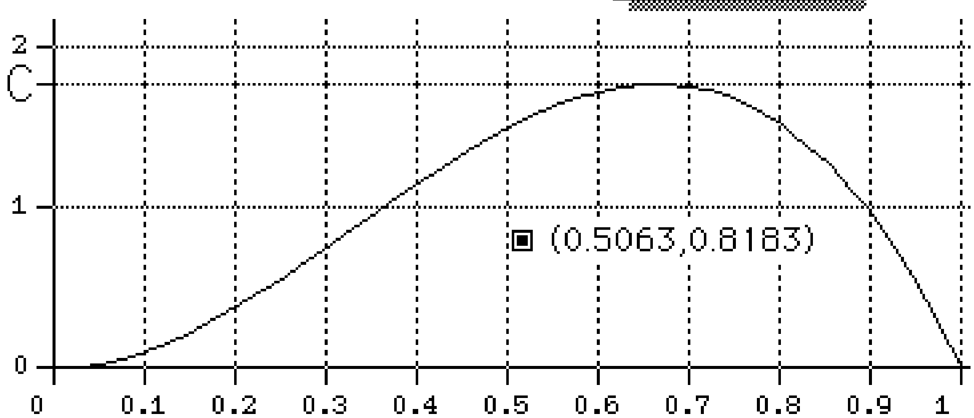
The next 2 uniformly-distributed random numbers are

$R_1 = 0.5063$, $R_2 = 0.4603$

$X = 0.5063$, $Y = R_2C = 0.8183$

$f(X) = 1.5187 > Y$ **ACCEPT!**

again, the point (x,y) is under the density curve!



0.5

The first 4 random numbers having the desired BETA distribution are, therefore,

0.4876
0.8147
0.7233
~~0.1129~~ *rejected*
0.5063