Efficiency of the Revised Simplex Method

Which version
- "ordinary" simplex method
- "revised" simplex method
requires the least computational effort?

Computational effort per pivot depends upon the problem parameters

\[ n = \text{# columns of } A \]
\[ m = \text{# constraints} \]
\[ d = \text{density of } A \text{ (% nonzero elements)} \]
Assume that, in the ordinary simplex tableau, previous pivots have increased the density such that we cannot make good use of sparse matrix techniques.

Let's count the number of multiplications & divisions per pivot.

Consider the operations in a pivot in row \( r \), column \( s \):

\[
\begin{bmatrix}
\hat{C}^1 & \hat{C}^2 & \ldots & \hat{C}^s & \ldots & \hat{C}^n & \hat{Z} \\
\hat{A}_1^1 & \hat{A}_1^2 & \ldots & \hat{A}_1^s & \ldots & \hat{A}_1^n & \hat{b}_1 \\
\hat{A}_2^1 & \hat{A}_2^2 & \ldots & \hat{A}_2^s & \ldots & \hat{A}_2^n & \hat{b}_2 \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots \\
\hat{A}_r^1 & \hat{A}_r^2 & \ldots & \hat{A}_r^s & \ldots & \hat{A}_r^n & \hat{b}_r \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots \\
\hat{A}_m^1 & \hat{A}_m^2 & \ldots & \hat{A}_m^s & \ldots & \hat{A}_m^n & \hat{b}_m \\
\end{bmatrix}
\]
> **Ordinary Simplex Method**

Pivoting in full tableau, with 100% density

> **Revised Simplex Method**

Explicit basis inverse maintained, and density less than 100%

> **Comparison of Algorithms**

<table>
<thead>
<tr>
<th>Ordinary Simplex Method</th>
<th>Operation Count</th>
<th>per iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(x and ÷)</td>
<td></td>
</tr>
<tr>
<td>Minimum Ratio Test (pivot row selection)</td>
<td>m divisions</td>
<td></td>
</tr>
</tbody>
</table>

**Pivot:**

- Divide row r by $\tilde{A_r}$ (need not divide in basic columns)
  - n-m divisions
For \( i=1,2,...,m+1, \ i \neq r, \)

add \(-\hat{A}_i\) times row \( r \) to row \( i \)

(only necessary to compute elements in nonbasic columns,)

\((n-m)\) multiplications

per each of \( m \) rows

---

**Ordinary Simplex Method**

Total number of multiplications & divisions:

\[
N_S = m + (n-m) + m(n-m)
\]

\[
= m + n + mn - m^2
\]

per iteration.
Revised Simplex Method

Operation Count
(x and ÷)
per iteration

- Pricing each of \((n-m)\) nonbasic columns
  (selecting pivot column)

  \(\tilde{c}^j = \pi A^j\)

  \((dm)\) multiplications per each of
  \((n-m)\) columns

- Computing substitution rates
  \(\tilde{A}^s = (A^B)^{-1} A^j\)
  (computing pivot column)

  \(dm\) multiplications per each of \(m\) rows

- Minimum ratio test (pivot row selection)

  \(m\) divisions
Pivot (update of basis inverse matrix, rhs, & \( \pi \))

- divide row \( r \) of \((A^B)^{-1} \& b\) by pivot element (m+1) divisions

- For \( i = 0, 1, 2, \ldots m \) (i = r):
  Add multiple of row \( r \) to row \( i \)

  \((m+1)\) multiplications per each of \( m \) rows

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Revised Simplex Method

Total number of multiplications \& divisions:

\[ N_R = dm(n-m) + dm^2 + m + (m+1) + (m+1)n \]

\[ = dmn + m^2 + 3m + 1 \]

per iteration.
Comparison of Algorithms

Multiplications & Divisions per iteration:

Ordinary Simplex \[ N_S = m + n + mn - m^2 \]

Revised Simplex \[ N_R = dmn + m^2 + 3m + 1 \]

Under what conditions is the revised simplex method more efficient than the ordinary simplex method?

That is, when is \( N_R < N_S \)?

\[ N_R < N_S \]
\[ \Rightarrow dmn + m^2 + 3m + 1 < m + n + mn - m^2 \]
\[ \Rightarrow dmn < mn + n - 2m^2 - 2m - 1 \]
\[ \Rightarrow d < 1 - \frac{2m}{n} + \frac{1}{m} - \frac{2}{n} - \frac{1}{mn} \approx 1 - \frac{2m}{n} \]

negligible
So the revised simplex method is more efficient than the ordinary simplex method when the density of the coefficient matrix $A$ satisfies:

$$d < 1 - 2\frac{m}{n}$$

For example:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$1 - 2\frac{m}{n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>50</td>
<td>60%</td>
</tr>
<tr>
<td>100</td>
<td>1000</td>
<td>80%</td>
</tr>
<tr>
<td>100</td>
<td>10000</td>
<td>98%</td>
</tr>
</tbody>
</table>

If $m=10$ & $n=50$, then the revised simplex method is more efficient if the density is less than about 60%.

$$N_S = m + n + mn - m^2$$
$$N_R = dm + m^2 + 3m + 1$$

For large LP problems in the "real world", the density is typically no more than 5%.

If $m=100$ and $n=1000$, $N_S = 91100$.

<table>
<thead>
<tr>
<th>$d=1%$</th>
<th>$d=5%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_R$</td>
<td>11301</td>
</tr>
<tr>
<td>$N_R/N_S$</td>
<td>0.124</td>
</tr>
</tbody>
</table>