A fixed sum of money $F$ is to be allocated among $n$ investments, each of which has a known history of returns during the previous $p$ periods.

$$f_{ik} = \text{return per dollar invested in investment } ^#i \text{ during period } k, \quad i=1,\ldots,n; \quad k=1,\ldots,p$$

$$X_i = \text{amount of money to be allocated to investment } ^#i$$
**EXAMPLE**  

\[ n=3, \ p=5 \]

<table>
<thead>
<tr>
<th>Investment</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>10%</td>
<td>4%</td>
<td>12%</td>
<td>13%</td>
<td>6%</td>
</tr>
<tr>
<td>#2</td>
<td>6%</td>
<td>9%</td>
<td>6%</td>
<td>5%</td>
<td>9%</td>
</tr>
<tr>
<td>#3</td>
<td>17%</td>
<td>1%</td>
<td>11%</td>
<td>19%</td>
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</tr>
</tbody>
</table>

\( r_{ik} \)

**Annual Return**

\[ F = $10,000 \text{ available for investment} \]

---

Let

\[ x_i = \text{amount of money to be allocated to investment } i \]

Assuming that past history is indicative of future performance, the expected annual return will be

\[ E = \sum_{i=1}^{n} e_i x_i \]

where

\[ e_i = \frac{1}{p} \sum_{k=1}^{p} r_{ik} \]

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Expected Annual Returns

\[
\begin{align*}
e_1 &= \frac{1}{5} (0.10 + 0.04 + 0.12 + 0.13 + 0.06) = 0.09 \\
e_2 &= \frac{1}{5} (0.06 + 0.09 + 0.06 + 0.05 + 0.09) = 0.07 \\
e_3 &= \frac{1}{5} (0.17 + 0.10 + 0.11 + 0.19 + 0.02) = 0.10
\end{align*}
\]

The expected total annual return from the investments will be

\[
E = 0.09 x_1 + 0.07 x_2 + 0.10 x_3
\]

If we wish to maximize the expected return, then we would invest the total available funds in investment \#3, which has the highest expected return.

Looking at the past history, however, we see a greater variability in the return provided by investment \#3:

<table>
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<tr>
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</tbody>
</table>

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The Variance of the total annual return, based upon past performance, is

\[
V = \frac{1}{p} \sum_{k=1}^{p} \left[ \sum_{i=1}^{n} r_{ik}x_i - E \right]^2
\]

\[
\text{total return in year } k \text{ if you had invested } x
\]

\[
\text{expected return per year}
\]

Substitute \( E = \sum_{i=1}^{n} e_i x_i \) into

\[
V = \frac{1}{p} \sum_{k} \left( \sum_{i} r_{ik} x_i - E \right)^2
\]

\[
= \frac{1}{p} \sum_{k} \left( \sum_{i} [r_{ik} - e_i] x_i \right)^2
\]

\[
= \frac{1}{p} \sum_{k} \sum_{i} \sum_{j} (r_{ik} - e_i) (r_{jk} - e_j) x_i x_j
\]

\[
V = \sum_{i} \sum_{j} \sigma_{ij}^2 x_i x_j \text{ where } \sigma_{ij}^2 = \frac{1}{p} \sum_{k} (r_{ik} - e_i) (r_{jk} - e_j)
\]

\[
= \frac{1}{p} \sum_{k} r_{ik} r_{jk} - \frac{1}{p^2} \left( \sum_{k} r_{ik} \right) \left( \sum_{k} r_{jk} \right)
\]
Thus, the variance of the total return is the quadratic function \( x^T C x \)
where \( C \) is the covariance matrix with entries \( \sigma_{ij}^2 \)

In our example, the covariance matrix is
\[
C = \begin{bmatrix}
12 & -5.6 & 23 \\
-5.6 & 2.8 & -12 \\
23 & -23 & 55.2
\end{bmatrix} \times 10^{-4}
\]

If we invest all $10,000 in #3 in order to maximize the expected return, i.e., \( x = (0,0,10^4) \) the variance of the annual return will be
\[
x^T C x = 55.2 \times 10^{-4} \times 10^4 \times 10^4 = 55.2 \times 10^4
\]
i.e., the standard deviation will be $743, or 74% of the expected return ($1000)! ... a "risky" investment.
Suppose that we are "risk-averse" and are satisfied with an 8% annual rate of return, but wish to minimize the variance of the total return.

Then we must solve:

\[
\text{Minimize } x^T \begin{bmatrix} 12 & -5.6 & 23 \\ -5.6 & 2.8 & -12 \\ 23 & -23 & 55.2 \end{bmatrix} x \\
\text{subject to } x_1 + x_2 + x_3 \leq 10000 \\
0.09x_1 + 0.07x_2 + 0.10x_3 \geq 800 \\
x_j \geq 0, j=1,2,3
\]

Optimal solution: \(x = (5000, 5000, 0)\)

i.e., invest half of the total in each of investments \(*1 & *2\), and nothing in \(*3\) which yields the greatest expected return!

The expected return will be 8% ($800), with a variance of 0.009\times10^6, \text{i.e., a standard deviation of } $94.87.