Hildeth & D'Espo Algorithm for Quadratic Programming

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Hildeth & D'Espo

A Cyclic Coordinate Search Method applied to the QP Dual Problem:

PRIMAL

Minimize $\frac{1}{2} x^T Q x + c^T x$

subject to $Ax \geq b$

DUAL

Maximize $e^T\lambda + \frac{1}{2}\lambda^T D \lambda$ $-\frac{1}{2} c^T Q^{-1} c$

subject to $\lambda \geq 0$

constant

where

$$\begin{cases} e = b + AQ^{-1}c \\ D = -AQ^{-1}A^T \end{cases}$$

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$$\boxed{\begin{array}{c}\text{Maximize } e^T\lambda + \frac{1}{2}\lambda^T D\,\lambda \\ \text{subject to } \lambda \geq 0\end{array}}$$

KKT Optimality Conditions

$$\boxed{\begin{array}{c}\nabla\hat{L}(\lambda) = D\lambda + e \leq 0 \\ \lambda_i \frac{\partial\hat{L}}{\partial\lambda_i} = 0\end{array}}$$

nonpositive, not nonnegative, because objective is MAX

"complementary slackness"

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A Cyclic Coordinate Search Method applied to the QP Dual Problem:

Step 0 : Select an initial λ° , e.g., $\lambda_i^\circ=0$

Step 1 : Let i=1 and $\lambda = \lambda^\circ$

Step 2 : Search for maximum in direction parallel to the λ_i-axis by fixing λ_j , $j\neq i$, and solving

$$\boxed{\frac{\partial\hat{L}}{\partial\lambda_i} = 0 \text{ for } \lambda_i}$$

a linear equation

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Step 3 : If $\lambda_i<0$, then fix $\lambda_i=0$.

$\boxed{\lambda_i \dfrac{\partial \hat{L}}{\partial \lambda_i} = 0} \Rightarrow$ If the unconstrained maximum of $\hat{L}$ (with respect to λ_i alone) is at $\lambda_i<0$, then the constrained max is at $\lambda_i=0$.

Step 4 : Increment i. If $i \leq n$, go to step 2.

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Step 5 : If $\lambda \neq \lambda^\circ$, then let $\lambda^\circ = \lambda$, and go to step 1. (The current λ will satisfy the KKT conditions for the QP dual.)

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☞ Example #1

☞ Example #2

☞ Example #3 (portfolio problem)

EXAMPLE

Minimize $\frac{1}{2}x_1^2 + \frac{1}{2}x_1^2 - 2x_1 - 2x_2$

subject to $\begin{cases} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{cases}$

that is, Minimize $\frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ -2 \end{bmatrix}^T x$

subject to $\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x \ge \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

Dual QP Problem

$$\text{Maximize} \begin{bmatrix}1\\1\\-2\\-2\end{bmatrix}^T \lambda + \frac{1}{2}\lambda^T \begin{bmatrix}1&0&1&0\\0&-1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\lambda$$

subject to $\lambda \ge 0$

$$\left\{\begin{array}{l} \text{Maximize } \lambda_1+\lambda_2-2\lambda_3-2\lambda_4 \\ \qquad -\frac{1}{2}\left[\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2\right]+\lambda_1\lambda_3+\lambda_2\lambda_4 \\ \text{subject to } \lambda_1\ge 0,\ \lambda_2\ge 0,\ \lambda_3\ge 0,\ \lambda_4\ge 0 \end{array}\right.$$

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KKT Conditions

$$\boxed{\begin{array}{l}\nabla\hat{L}(\lambda) \le 0 \\ \lambda_i \dfrac{\partial\hat{L}}{\partial\lambda_i}=0\end{array}}$$

$$\nabla\hat{L}(\lambda) = D\lambda + e = \begin{bmatrix}-\lambda_1+\lambda_3+1\\ -\lambda_2+\lambda_1+1\\ \lambda_1-\lambda_3-2\\ \lambda_2-\lambda_4-2\end{bmatrix}$$

$$\boxed{\lambda_i \frac{\partial\hat{L}}{\partial\lambda_i}=0} \left\{\begin{array}{l}\lambda_1(-\lambda_1+\lambda_3+1)=0\\ \lambda_2(-\lambda_2+\lambda_4+1)=0\\ \lambda_3(\lambda_1-\lambda_3-2)=0\\ \lambda_4(\lambda_2-\lambda_4-2)=0\end{array}\right.$$

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Start with $\lambda = \lambda^0 = (0,0,0,0)$

i=1 Solve $-\lambda_1 + \lambda_3 + 1 = 0$ for λ_1, with $\lambda_3 = 0$, to get $\lambda_1 = 1$, so that $\lambda = (1,0,0,0)$

i=2 Solve $-\lambda_2 + \lambda_4 + 1 = 0$ for λ_2, with $\lambda_4 = 0$, to get $\lambda_2 = 1$, so that $\lambda = (1,1,0,0)$

i=3 Solve $\lambda_1 - \lambda_3 - 2 = 0$ for λ_3, with $\lambda_1 = 1$, to get $\lambda_3 = -1 < 0$. The maximizing λ_3 is therefore 0, so that $\lambda = (1,1,0,0)$

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i=4 Solve $\lambda_2 - \lambda_4 - 2 = 0$ for λ_4, with $\lambda_2 = 1$, to get $\lambda_4 = -1 < 0$. The maximizing λ_4 is therefore 0, so that $\lambda = (1,1,0,0)$

(end of cycle)

Since $(1,1,0,0) = \lambda \neq \lambda^0 = (0,0,0,0)$, i.e., λ was changed during the cycle, repeat the cycle.

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$\lambda^0 = \lambda = (1,1,0,0)$

i=1 Solve $-\lambda_1 + \lambda_3 + 1 = 0$ for λ_1, with $\lambda_3 = 0$,
to get $\lambda_1 = 1$, so that $\lambda = (1,0,0,0)$

i=2 Solve $-\lambda_2 + \lambda_4 + 1 = 0$ for λ_2, with $\lambda_4 = 0$,
to get $\lambda_2 = 1$, so that $\lambda = (1,1,0,0)$

i=3 Solve $\lambda_1 - \lambda_3 - 2 = 0$ for λ_3, with $\lambda_1 = 1$,
to get $\lambda_3 = -1 < 0 \Rightarrow \lambda_3 = 0$ so that $\lambda = (1,1,0,0)$

i=4 Solve $\lambda_2 - \lambda \ - 2 = 0$ for λ_4, with $\lambda_2 = 1$,
to get $\lambda_4 = -1 < 0 \Rightarrow \lambda_4 = 0$ so that $\lambda = (1,1,0,0)$

(end of cycle)

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Since $\lambda = \lambda^0 = (1,1,0,0)$, i.e., λ was unchanged during the cycle, the algorithm has converged, and $\lambda^* = (1,1,0,0)$ is the optimum of the QP dual problem.

recovery of primal optimal variables

$$x^*(\lambda^*) = Q^{-1}\left[A^T\lambda^* - c\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\lambda^* - \begin{bmatrix} -2 \\ -2 \end{bmatrix}\right)$$

$$\Longrightarrow \begin{cases} x_1^* = -\lambda_1^* + \lambda_3^* + 2 = 1 \\ x_2^* = -\lambda_2^* + \lambda_4^* + 2 = 1 \end{cases}$$

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EXAMPLE Consider the convex QP problem

Minimize

$$2\,x_1^2 + x_2^2 - 2x_1x_2 - 4x_1 - 6x_2$$

subject to
$$\begin{cases} x_1 + x_2 \leq 8 \\ -x_1 + 2x_2 \leq 10 \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases}$$

that is,

$$\text{Min } \frac{1}{2}x^T\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}x + \begin{bmatrix} -4 & -6 \end{bmatrix}x$$

$$\text{subject to } \begin{bmatrix} -1 & -1 \\ +1 & -2 \\ +1 & 0 \\ 0 & +1 \end{bmatrix} x \geq \begin{bmatrix} -8 \\ -10 \\ 0 \\ 0 \end{bmatrix}$$

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Quadratic terms:

$$D = -A\,Q^{-1}\,A^T = -\begin{bmatrix} -1 & -1 \\ +1 & -2 \\ +1 & 0 \\ 0 & +1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

Computation of QP Dual Objective Function

$$= \begin{bmatrix} -5/2 & -2 & 1 & 3/2 \\ -2 & -5/2 & 1/2 & 3/2 \\ 1 & 1/2 & -1/2 & -1/2 \\ 3/2 & 3/2 & -1/2 & 1 \end{bmatrix}$$

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Linear terms:

$$e = b + A\,Q^{-1}c \;=\; \begin{bmatrix} -8 \\ -10 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ +1 & -2 \\ +1 & 0 \\ 0 & +1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ -10 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 13 \\ 11 \\ -5 \\ -8 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -5 \\ -8 \end{bmatrix}$$

Computation of QP Dual Objective Function

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$$\boxed{\begin{array}{c} \text{Maximize } \ \frac{1}{2}\lambda^{T}D\,\lambda \ + \ e^{T}\lambda \\ \text{subject to } \lambda \geq 0 \end{array}}$$

QP Dual Problem

$$\text{Maximize} \quad 1/2\,\{ -5/2\lambda_1^2 - 5/2\lambda_2^2 + 1/2\lambda_3^2 + \lambda_4^2$$
$$- 4\,\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 3\lambda_1\lambda_4$$
$$+ 2\lambda_2\lambda_3 + 3\lambda_2\lambda_4 - \lambda_3\lambda_4 \}$$
$$+ 5\lambda_1 + \lambda_2 - 5\lambda_3 - 8\lambda_4$$

$$\text{subject to} \qquad \lambda \geq 0$$

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KKT Conditions

$$D\lambda + e \le 0$$

$$\lambda\left[D\lambda + e\right] = 0$$

$$\lambda \ge 0$$

i.e.,

$$\begin{cases} \lambda_1\left[-\tfrac{5}{2}\lambda_1 - 2\lambda_2 + \lambda_3 + \tfrac{3}{2}\lambda_4 + 5\right] = 0 \\ \lambda_2\left[-2\lambda_1 - \tfrac{5}{2}\lambda_2 + \tfrac{1}{2}\lambda_3 + \tfrac{3}{2}\lambda_4 + 1\right] = 0 \\ \lambda_3\left[\lambda_1 + \tfrac{1}{2}\lambda_2 - \tfrac{1}{2}\lambda_3 - \tfrac{1}{2}\lambda_4 - 5\right] = 0 \\ \lambda_4\left[\tfrac{3}{2}\lambda_1 + \tfrac{3}{2}\lambda_2 - \tfrac{1}{2}\lambda_3 - \lambda_4 - 8\right] = 0 \end{cases}$$

$$\lambda_1 \ge 0,\ \lambda_2 \ge 0,\ \lambda_3 \ge 0,\ \lambda_4 \ge 0$$

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Let $\lambda = (0,0,0,0)$

Either $\lambda_1 = 0$

or $\lambda_1 = \tfrac{2}{5}\left[5 - 2\lambda_2 + \lambda_3 + \tfrac{3}{2}\lambda_4\right] = 2$

Let $\lambda_1 = 2$

$\lambda = (2,0,0,0)$

Either $\lambda_2 = 0$

or $\lambda_2 = \tfrac{2}{5}\left[1 - 2\lambda_1 + \tfrac{1}{2}\lambda_3 + \tfrac{3}{2}\lambda_4\right] = -\tfrac{6}{5}$

Let $\lambda_2 = 0$

$\lambda = (2,0,0,0)$

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Either $\lambda_3 = 0$

or $\lambda_3 = 2\left[-5 + \lambda_1 + \frac{1}{2}\lambda_2 - \frac{1}{4}\lambda_4\right] = -6$

Let $\lambda_3 = 0$

$\lambda = (2,0,0,0)$

Either $\lambda_4 = 0$

or $\lambda_4 = \left[-8 + \frac{3}{2}\lambda_1 + \frac{3}{2}\lambda_2 - \frac{1}{2}\lambda_3\right] = -5$

Let $\lambda_4 = 0$

$\lambda = (2,0,0,0)$

$\lambda = (2,0,0,0) \neq \lambda^o = (0,0,0,0)$

Therefore, repeat the cycle.

(This will leave λ unchanged, so that $\lambda^* = (2,0,0,0)$ satisfies the KKT conditions and is therefore optimal for the QP dual problem.)

The primal optimal solution is then recovered by

$$x^*(\lambda^*) = Q^{-1}\left[A^T\lambda^* - c\right]$$

$$x^*(\lambda^*) = Q^{-1}\left[A^T\lambda^* - \mathbf{c}\right] \quad \text{where} \quad \lambda^* = (2,0,0,0)$$

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} -\lambda_1^* - 1/2\,\lambda_2^* + 1/2\,\lambda_3^* + 1/2\,\lambda_4^* + 5 \\ -3/2\,\lambda_1^* - 3/2\,\lambda_2^* + 1/2\,\lambda_3^* + \lambda_4^* + 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

EXAMPLE

"Optimal Portfolio Problem"

$$\textbf{Minimize } x^T \begin{bmatrix} 12 & -5.6 & 23 \\ -5.6 & 2.8 & -12 \\ 23 & -23 & 55.2 \end{bmatrix} x$$

subject to

$$x_1 + x_2 + x_3 \leq 10000$$

$$0.09x_1 + 0.07x_2 + 0.10\ x_3 \geq 800$$

$$x_j \geq 0,\ j=1,2,3$$

Hessian Matrix of Objective

```
12    ¯5.6  23
¯5.6   2.8 ¯12
23   ¯12    55.2
```

Linear Cost Coefficients

```
i    1  2  3
C[i] 0  0  0
```

Constraint Coefficients

```
  1      1      1    ≤  10000
¯0.09  ¯0.07  ¯0.1   ≤   ¯800
```

plus nonnegativity constraints: X ≥ 0

```
The QP dual is
               Max (E+.×L) + .5×(⍉L)+.×D+.×L
               subject to  L≥0
where L is the vector of dual variables
```

The D matrix (Hessian of dual objective) is

```
¯43.987252  3.3512039  8.2294617  32.174220  3.5835694
  3.351203 ¯0.2553307 ¯0.6283286  ¯2.450212 ¯0.2726628
  8.229461 ¯0.6283286 ¯1.8696883  ¯5.864022 ¯0.4957507
 32.174220 ¯2.4502124 ¯5.8640226 ¯23.618980 ¯2.6912181
  3.583569 ¯0.2726628 ¯0.4957507  ¯2.691218 ¯0.3966005
```

The E vector (linear coefficients of dual objective) is

```
¯10000 800 0 0 0
```

The Kuhn-Tucker conditions for the dual are:

```
L × ( D+.×L + E ) = 0,
L≥0
```

iteration	Lambda				
1	0.000E0	3.133E3	0.000E0	0.000E0	0.000E0
2	1.137E1	3.282E3	0.000E0	0.000E0	0.000E0
3	2.273E1	3.432E3	0.000E0	0.000E0	0.000E0
4	3.410E1	3.581E3	0.000E0	0.000E0	0.000E0
5	4.546E1	3.730E3	0.000E0	0.000E0	0.000E0
6	5.682E1	3.879E3	0.000E0	0.000E0	0.000E0
7	6.819E1	4.028E3	0.000E0	0.000E0	0.000E0
8	7.955E1	4.177E3	0.000E0	0.000E0	0.000E0
9	9.091E1	4.326E3	0.000E0	0.000E0	0.000E0
10	1.023E2	4.475E3	0.000E0	0.000E0	0.000E0
11	1.136E2	4.625E3	0.000E0	0.000E0	0.000E0
12	1.250E2	4.774E3	0.000E0	0.000E0	0.000E0
⋮	⋮	⋮	⋮	⋮	⋮
120	1.347E3	2.082E4	0.000E0	0.000E0	0.000E0
121	1.359E3	2.097E4	0.000E0	0.000E0	0.000E0
122	1.370E3	2.111E4	0.000E0	0.000E0	0.000E0
123	1.381E3	2.126E4	0.000E0	0.000E0	0.000E0
124	1.393E3	2.141E4	0.000E0	0.000E0	0.000E0
125	1.404E3	2.156E4	0.000E0	0.000E0	0.000E0

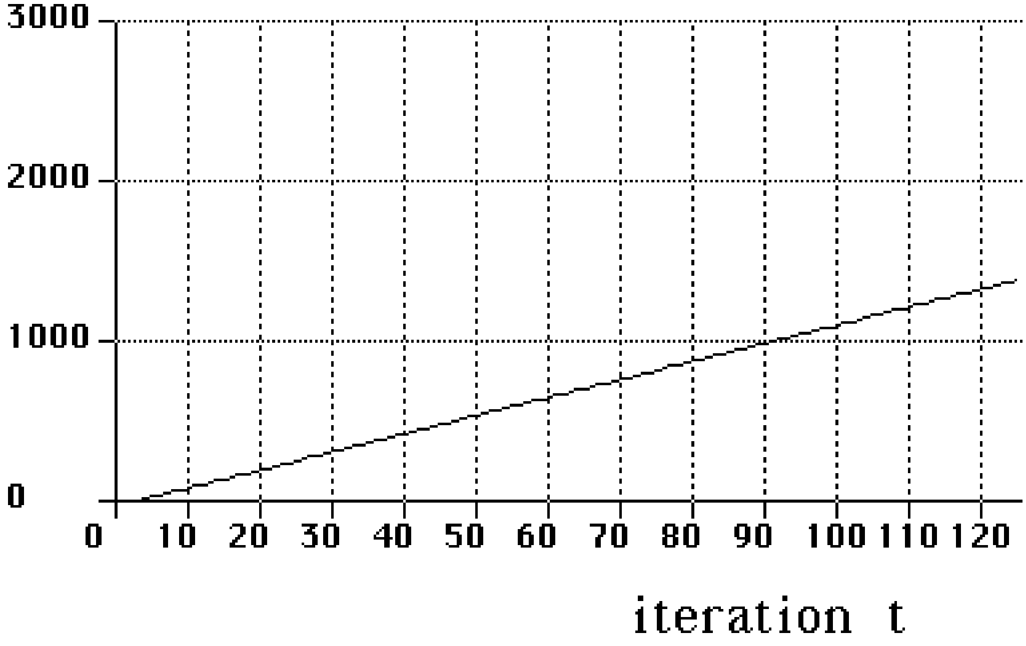
SOLUTION

(after 125 iterations, without converging!)

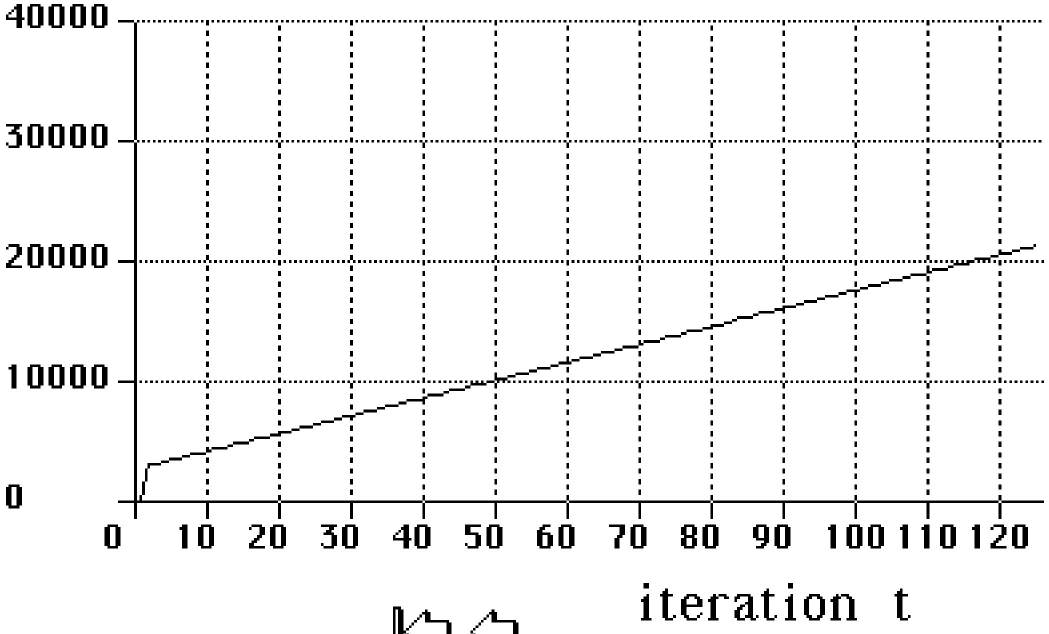
Portfolio Example

```
Primal Variables: x = 1992.79439 7655.682777 847.5071055
       Slack:     y = ¯495.9842719 5.684341886E¯13

Dual Variables:   Lambda = 1392.503289 21409.73162 0 0 0

Objective Function:  1256046.34
```

The optimal primal solution is (5000,5000,0)



λ_1^t



λ_2^t