

Example: Primal-Dual Interior Point Algorithm

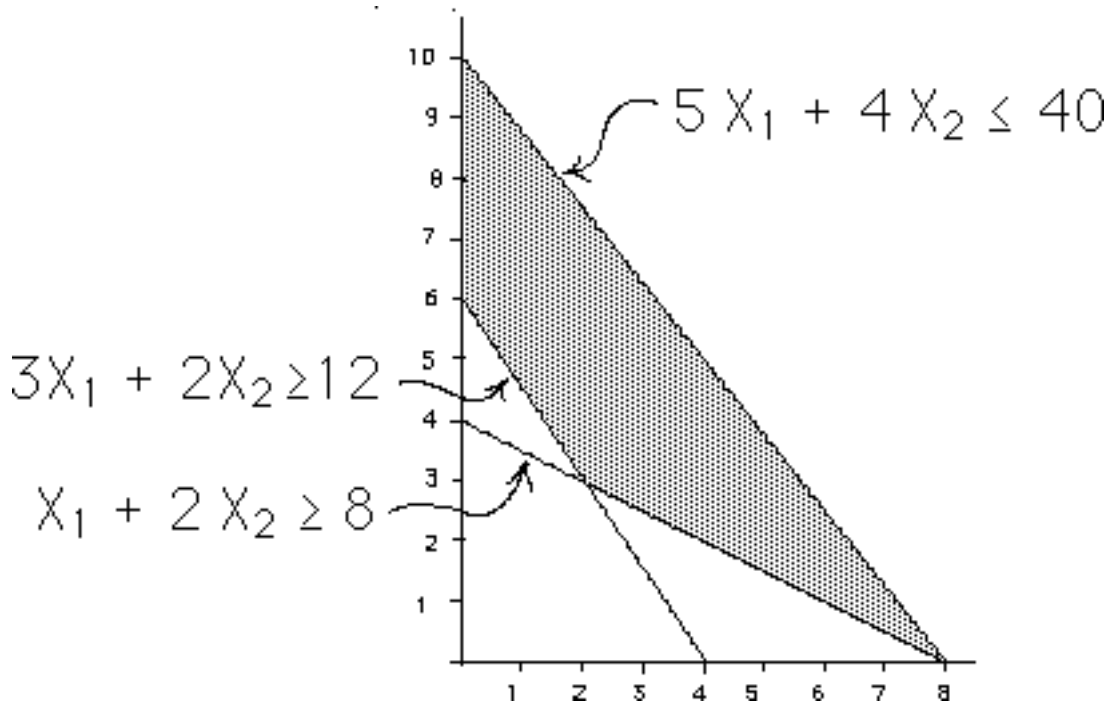


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Example

Minimize $6X_1 + 5X_2$
subject to

$$\begin{cases} 3X_1 + 2X_2 \geq 12 \\ X_1 + 2X_2 \geq 8 \\ 5X_1 + 4X_2 \leq 40 \\ X_1 \geq 0, X_2 \geq 0 \end{cases}$$



Convert constraints to equalities:

Minimize $6X_1 + 5X_2$

subject to

$$\begin{cases} 3X_1 + 2X_2 - X_3 & = 12 \\ X_1 + 2X_2 - X_4 & = 8 \\ 5X_1 + 4X_2 + X_5 & = 40 \end{cases}$$

$$X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, X_4 \geq 0, X_5 \geq 0$$

Dual Problem

Maximize $12 Y_1 + 8 Y_2 + 40 Y_3$
 subject to

$$\left\{ \begin{array}{l} 3 Y_1 + Y_2 + 5 Y_3 \leq 6 \\ 2 Y_1 + 2 Y_2 + 4 Y_3 \leq 5 \\ - Y_1 \leq 0 \\ \quad - Y_2 \leq 0 \\ \qquad Y_3 \leq 0 \end{array} \right.$$

$Y_1, Y_2, \& Y_3$ unrestricted in sign

Maximize $12 Y_1 + 8 Y_2 + 40 Y_3$
 subject to

*Dual in
equality form*

$$\left\{ \begin{array}{l} 3 Y_1 + Y_2 + 5 Y_3 + Z_1 = 6 \\ 2 Y_1 + 2 Y_2 + 4 Y_3 + Z_2 = 5 \\ - Y_1 + Z_3 = 0 \\ \quad - Y_2 + Z_4 = 0 \\ \qquad Y_3 + Z_5 = 0 \end{array} \right.$$

$Y_1, Y_2, \& Y_3$ unrestricted in sign

$Z_1 \geq 0, Z_2 \geq 0, Z_3 \geq 0, Z_4 \geq 0, Z_5 \geq 0$

Complementary Slackness Theorem:

If X is feasible in the primal, and
 Y, Z is feasible in the dual problem,
and if

$$X_i Z_i = 0 \text{ for } i=1,2,3,4, \text{ \& } 5$$

*i.e., for each i , either X_i or Z_i (or both)
must be zero*

Then X, Y, Z are optimal.

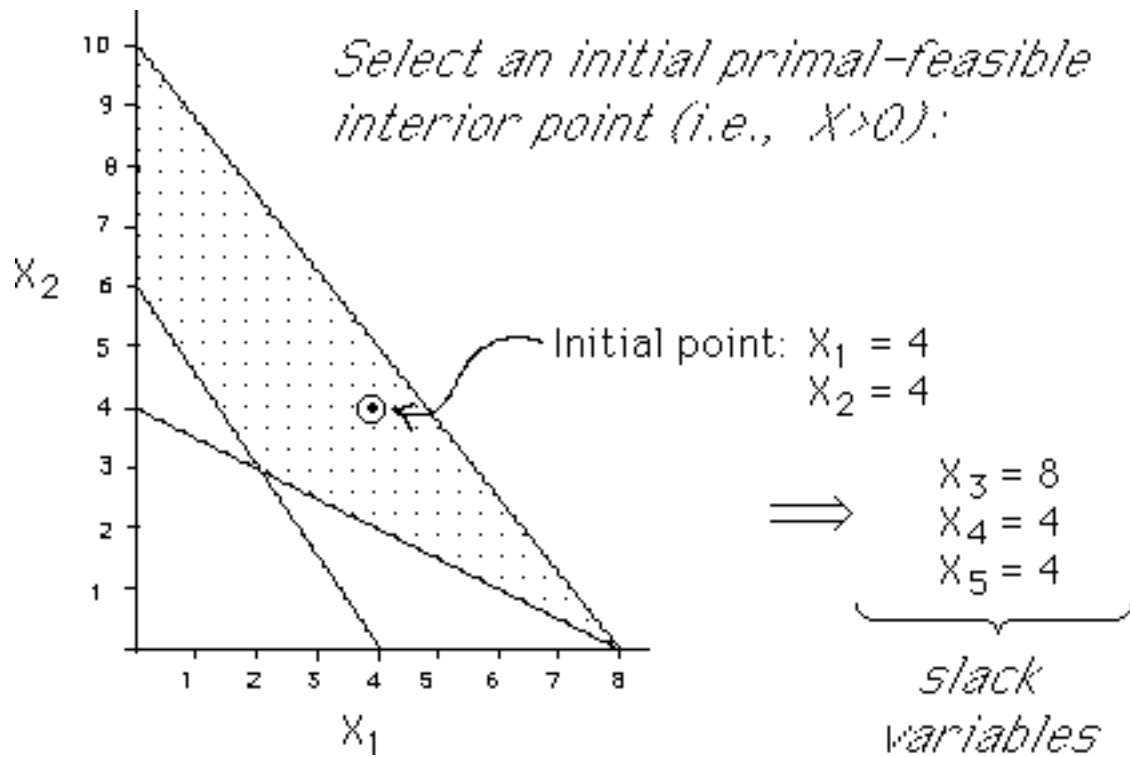
Algorithm Parameters

0.001 = Epsilon (convergence tolerance)

0.01 = alpha (shrinkage factor for μ)

0.995 = tau (stepsize factor)

10 = max number of iterations



Select an initial dual-feasible solution with $Z > 0$

$$\left\{ \begin{array}{rcl} 3 Y_1 + Y_2 + 5 Y_3 + Z_1 & & = 6 \\ 2 Y_1 + 2 Y_2 + 4 Y_3 + Z_2 & & = 5 \\ - Y_1 & & + Z_3 = 0 \\ & - Y_2 & + Z_4 = 0 \\ & & Y_3 + Z_5 = 0 \end{array} \right.$$

$Y_1, Y_2, \& Y_3$ unrestricted in sign

$$Z_1 \geq 0, Z_2 \geq 0, Z_3 \geq 0, Z_4 \geq 0, Z_5 \geq 0$$

Choose initial dual solution: $Y_1 = Y_2 = +1, Y_3 = -1$

$$Z_1 = 7, Z_2 = 5, Z_3 = Z_4 = Z_5 = +1$$

Iteration # 1

cx = 44, by = -20, +/xz = 64, gap = 64

Infeasibilities:

primal (dp): 0 0 0
 dual (dd): 0 0 0 0 0

i	x	Δx	x	z	Δz	z	xz
1	4	-1.71974	0.0390203	7	-0.790452	5.62717	0.219574
2	4	0.851613	5.96147	5	-2.86452	0.025	0.149037
3	8	-3.456	0.04	1	1.032	2.79234	0.111694
4	4	-0.0165161	3.96196	1	2.20413	4.82806	19.1286
5	4	5.19226	15.959	1	0.901935	2.56645	40.9581

j	yo	Δy	y
1	1	1.032	2.79234
2	1	2.20413	4.82806
3	-1	-0.901935	-2.56645

Stepsize: (primal) 2.30324
 (dual) 1.73677

updated mu= 0.60567
 gap= 60.567

Iteration # 2

cx = 30.0415, by = -30.5255, +/xz = 60.567, gap = 60.567

Infeasibilities:

primal (dp): 0 4.44089E-15 -7.10543E-15
 dual (dd): 0 0 0 0 0

i	xo	Δx	x
1	0.0390203	0.114609	7.30237
2	5.96147	-0.080621	0.852094
3	0.04	0.182584	11.6113
4	3.96196	-0.0466335	1.00655
5	15.959	-0.250559	0.0797951

i	zo	Δz	z	xz
1	5.62717	-6.63311	0.0281358	0.205458
2	0.025	0.0769355	0.0899415	0.0766386
3	2.79234	-0.396521	2.45764	28.5364
4	4.82806	-4.61836	0.929686	0.935779
5	2.56645	-2.48821	0.466148	0.0371964

j	yo	Δy	y
1	2.79234	-0.396521	2.45764
2	4.82806	-4.61836	0.929686
3	-2.56645	2.48821	-0.466148

Stepsize: (primal) 63.3752
 (dual) 0.844103

updated mu= 0.297914
 gap= 29.7914

Iteration # 3

cx = 48.0747, by = 18.2832, +/xz = 29.7914, gap = 29.7914

Infeasibilities:

primal (dp): -3.1493E-11 -2.64926E-11 -8.81215E-11
dual (dd): 8.88178E-16 0 0 0 0

i	x0	Δx	x
1	7.30237	-0.492754	0.0365118
2	0.852094	0.409379	6.88855
3	11.6113	-0.659504	1.88663
4	1.00655	0.326003	5.8136
5	0.0797951	0.826255	12.2633

i	z0	Δz	z	xz
1	0.0281358	0.0145597	0.0324661	0.0011854
2	0.0899415	0.216473	0.154324	1.06307
3	2.45764	-2.29239	1.77584	3.35036
4	0.929686	-0.934819	0.651655	3.78847
5	0.466148	-1.55949	0.00233074	0.0285825

j	y0	Δy	y
1	2.45764	-2.29239	1.77584
2	0.929686	-0.934819	0.651655
3	-0.466148	1.55949	-0.00233074

Stepsize: (primal) 14.7454
(dual) 0.297417

updated mu= 0.0823166
gap= 8.23166

Iteration # 4

cx = 34.6618, by = 26.4301, +/xz = 8.23166, gap = 8.23166

Infeasibilities:

primal (dp): 4.09543E-10 3.62401E-10 1.23953E-9
 dual (dd): 8.88178E-16 0 0 0 0

i	x0	Δx	x
1	0.0365118	2.36805	2.14121
2	6.88855	-4.4382	2.94393
3	1.88663	-1.77223	0.311487
4	5.8136	-6.50834	0.029068
5	12.2633	5.91252	17.5182

i	z0	Δz	z	xz
1	0.0324661	0.116391	0.448625	0.9606
2	0.154324	-0.0429454	0.000771622	0.0022716
3	1.77584	-0.0640448	1.54685	0.481824
4	0.651655	0.0920335	0.980724	0.0285077
5	0.00233074	0.00325799	0.0139798	0.244901

j	y0	Δy	y
1	1.77584	-0.0640448	1.54685
2	0.651655	0.0920335	0.980724
3	-0.00233074	-0.00325799	-0.0139798

Stepsize: (primal) 0.888788
 (dual) 3.57553

updated mu= 0.017181
 gap= 1.7181

Iteration # 5

cx = 27.5669, by = 25.8488, +/xz = 1.7181, gap = 1.7181

Infeasibilities:

primal (dp): 4.55458E-11 4.03055E-11 1.37852E-10

dual (dd): -1.77636E-15 0 0 0 0

i	x0	Δx	x
1	2.14121E0	-1.66109E-1	1.98864E0
2	2.94393E0	8.04421E-2	3.01781E0
3	3.11487E-1	-3.37443E-1	1.55744E-3
4	2.9068E-2	-5.22472E-3	2.42693E-2
5	1.75182E1	5.08776E-1	1.79855E1

i	z0	Δz	z	xz
1	4.48625E-1	-4.05798E-1	2.75444E-2	5.4776E-2
2	7.71622E-4	5.04339E-3	6.00495E-3	1.81218E-2
3	1.54685E0	1.84052E-1	1.73783E0	2.70657E-3
4	9.80724E-1	-2.13384E-1	7.59303E-1	1.84278E-2
5	1.39798E-2	-1.3405E-2	6.98989E-5	1.25717E-3

j	y0	Δy	y
1	1.54685	0.184052	1.73783
2	0.980724	-0.213384	0.759303
3	-0.0139798	0.013405	-0.0000698989

Stepsize: (primal) 0.918467
 (dual) 1.03766

updated mu= 0.000952893
 gap= 0.0952893

Iteration # 6

cx = 27.0209, by = 26.9256, +/xz = 0.0952893, gap = 0.0952893

Infeasibilities:

primal (dp): 3.96128E-12 3.10774E-12 1.09495E-11
 dual (dd): -8.88178E-16 0 0 0 0

i	xo	Δx	x
1	1.98864E0	1.08473E-2	2.00018E0
2	3.01781E0	-1.6781E-2	2.99997E0
3	1.55744E-3	-1.02002E-3	4.73056E-4
4	2.42693E-2	-2.27147E-2	1.21346E-4
5	1.79855E1	1.28874E-2	1.79992E1

i	zo
1	2.75444E-2
2	6.00495E-3
3	1.73783E0
4	7.59303E-1
5	6.98989E-5

j	yo	Δy	y
1	1.73783	0.0121683	1.75009
2	0.759303	-0.00937437	0.749863
3	-0.0000698989	0.0000169679	-0.0000528118

Stepsize: (primal) 1.0631
 (dual) 1.00703

updated mu= 0.0000307315
 gap= 0.00307315

Iteration # 7

cx = 27.0009, by = 26.9978, +/xz = 0.00307315, gap = 0.00307315

Infeasibilities:

primal (dp): -2.46914E-13 -1.88294E-13 -6.89226E-13
dual (dd): 0 8.88178E-16 0 0 0

i	x0	Δx	x
1	2.00018E0	-1.87543E-4	1.99998E0
2	2.99997E0	5.35786E-5	3.00003E0
3	4.73056E-4	-4.55472E-4	2.36528E-6
4	1.21346E-4	-8.03859E-5	3.82746E-5
5	1.79992E1	7.23401E-4	1.8E1

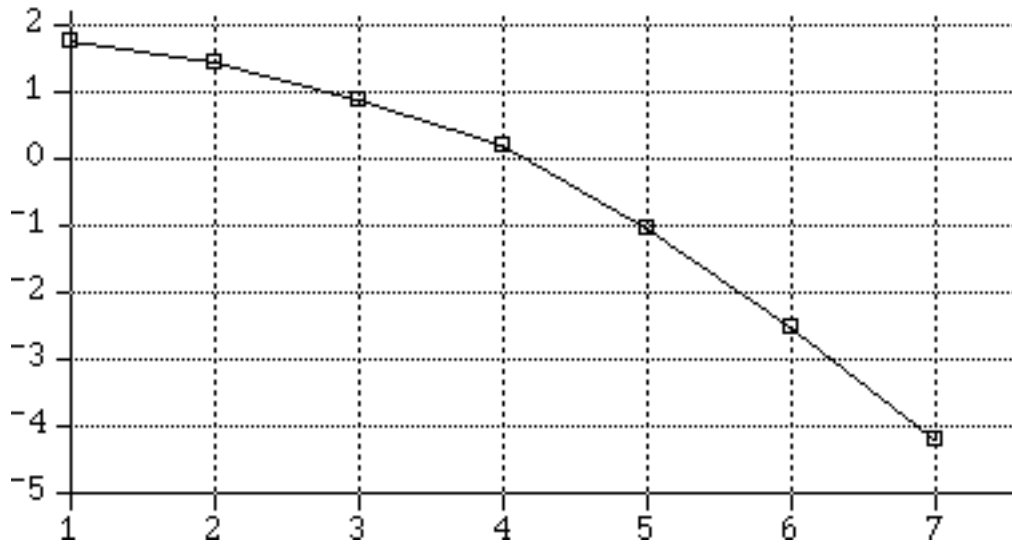
i	z0	Δz	z	xz
1	1.37722E-4	-1.22345E-4	1.19272E-5	2.38542E-5
2	3.09413E-4	-2.99175E-4	1.80171E-6	5.40519E-6
3	1.75009E0	-9.02812E-5	1.74999E0	4.13923E-6
4	7.49863E-1	1.37655E-4	7.50005E-1	2.87061E-5
5	5.28118E-5	-5.11065E-5	2.64059E-7	4.75306E-6

j	y0	Δy	y
1	1.75009E0	-9.02812E-5	1.74999E0
2	7.49863E-1	1.37655E-4	7.50005E-1
3	-5.28118E-5	5.11065E-5	-2.64059E-7

Stepsize: (primal) 1.03341
(dual) 1.0282

updated mu= 6.68578E-7
gap= 0.0000668578

Log (base 10) of gap vs iteration



Plot of X[2] vs X[1]

