Maximum Flow Problem

Find the maximum possible amount of flow in the network from the source $s$ to the sink $t$. 
Algorithm

Given: a network with designated source & sink, each arc having a capacity in each direction. (Capacity of arc \((i,j)\) need not equal that of \((j,i)\))

Step 0 Initially, let the flow in each arc be zero.

Step 1 Find any path from source to sink that has positive flow capacity (in direction of flow) for every arc in the path. If no such path exists, STOP.

(For example, try to construct a spanning tree, using only arcs with positive capacity.)

Step 2 Find the smallest arc capacity \(k\) on this path (the flow-augmenting path). Increase the flow in this path by \(k\).

Step 3 For each arc in the flow-augmenting path, reduce all capacities in the direction of the flow by the amount \(k\), and increase all capacities in the direction opposite the flow by \(k\).

Return to Step 1.

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Select flow-augmenting path 1-2-5-7. Smallest capacity on this path is 2.
Send 2 units of flow along the path. Update the arc capacities (forward & backward).

Find the next flow-augmenting path. Select 1-2-5-4-7. Smallest capacity on path is 1.

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Send 1 unit of flow along the path.
Update the arc capacities (forward & backward) along the path.

Find another flow-augmenting path, e.g., 1-3-6-7
Smallest capacity along path is 3.
Send 3 units of flow along the path.
Update the capacities along the path (forward & backward.)

Find another flow-augmenting path, e.g., 1-4-3-6-7.
Smallest capacity along the path is 2.
Send 2 units of flow along the path.

Update the capacities (forward & backward) along the path.

Find the next flow-augmenting path, e.g., 1-4-7.

Smallest capacity along this path is 2.
Send 2 units of flow along this path.
Update the capacities (forward & backward).

No flow-augmenting path can now be found.
Capacity across this "cut" is zero.
**Definition**

A cut of a network is a partition of the node set $N$ into 2 subsets, $N_1$ and $N_2$, such that

- $N = N_1 \cup N_2$,
- $N_1 \cap N_2 = \emptyset$,
- the source node is in $N_1$,
- the sink node is in $N_2$.

The capacity of the cut is $\sum \sum c_{ij}$ for $i \in N_1, j \in N_2$.

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**Example**

Consider the original arc capacities.

The flow in a network is bounded above by the capacity of any cut.
Capacity of this "cut" is 10
= maximum flow

**MAX-FLOW/MIN-CUT THEOREM**

The maximum flow in a network is equal to the capacity of the cut having the minimum cut capacity.