$M/E_k/1/N/N$ Queueing System

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A machine operator has responsibility for four semi-automatic machines.

While processing jobs, the machines require no attention from the operator.
When a job is complete, the operator must

	Average Time Req'd
1) Unload the old job	$1/\mu_1 = 15$ seconds
2) Load the new job	$1/\mu_2 = 20$ seconds
3) Restart the machine	$1/\mu_3 = 10$ seconds
Total	45 seconds

Assuming that the time for each of the three tasks has exponential distribution, we wish to compute

- Steadystate distribution of number of machines in operation
- Average utilization of machines

for jobs with exponentially-distributed processing time, where the mean is 5 minutes

Mean service time is $\sum_{i=1}^{3} 1/\mu_i = 45$ seconds

Variance of service time is $\sum_{i=1}^{3} \left(1/\mu_i\right)^2 = 725$

i.e., standard deviation is 26.925824 seconds, substantially less than 45, the standard deviation of exponential dist'n with mean 45 sec.

If $\mu_1 = \mu_2 = \mu_3 = \mu$, then the service time has Erlang-3 probability distribution.

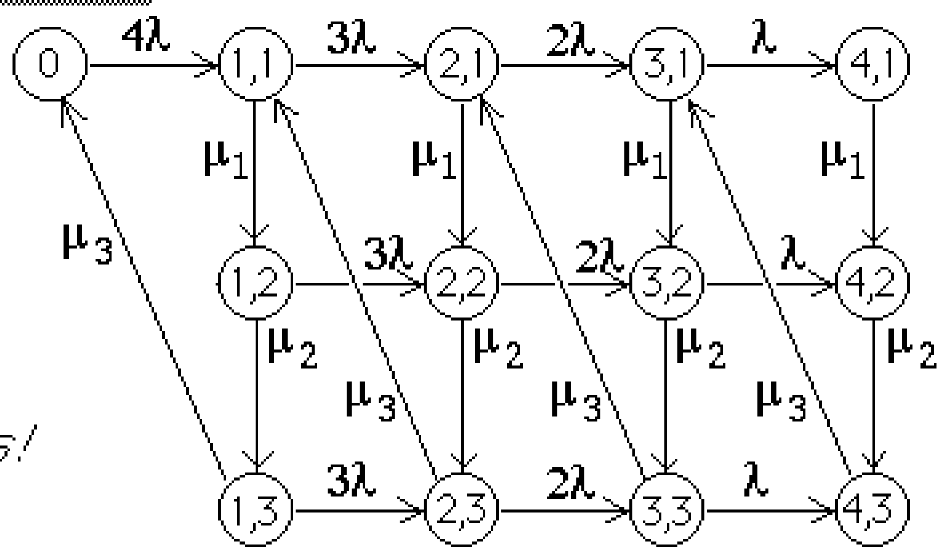
Continuous-Time Markov Chain

Define states:

(0) all machines in operation

(i,j) i machines out of operation with operator currently performing task j

Continuous-Time Markov Chain



Not a birth-death process!

The 13 States of the C-T Markov Chain

i	j	t
1	0	0
2	1	1
3	1	2
4	1	3
5	2	1
6	2	2
7	2	3
8	3	1
9	3	2
10	3	3
11	4	1
12	4	2
13	4	3

where i = state number,
j = # customers in system, and
t = task currently being performed

Transition Rate Matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	-0.8	0.8	0	0	0	0	0	0	0	0	0	0	0
2	0	-4.6	4	0	0.6	0	0	0	0	0	0	0	0
3	0	0	-3.6	3	0	0.6	0	0	0	0	0	0	0
4	6	0	0	-6.6	0	0	0.6	0	0	0	0	0	0
5	0	0	0	0	-4.4	4	0	0.4	0	0	0	0	0
6	0	0	0	0	0	-3.4	3	0	0.4	0	0	0	0
7	0	6	0	0	0	0	-6.4	0	0	0.4	0	0	0
8	0	0	0	0	0	0	0	-4.2	4	0	0.2	0	0
9	0	0	0	0	0	0	0	0	-3.2	3	0	0.2	0
10	0	0	0	0	6	0	0	0	0	-6.2	0	0	0.2
11	0	0	0	0	0	0	0	0	0	0	-4	4	0
12	0	0	0	0	0	0	0	0	0	0	0	-3	3
13	0	0	0	0	0	0	0	6	0	0	0	0	-6

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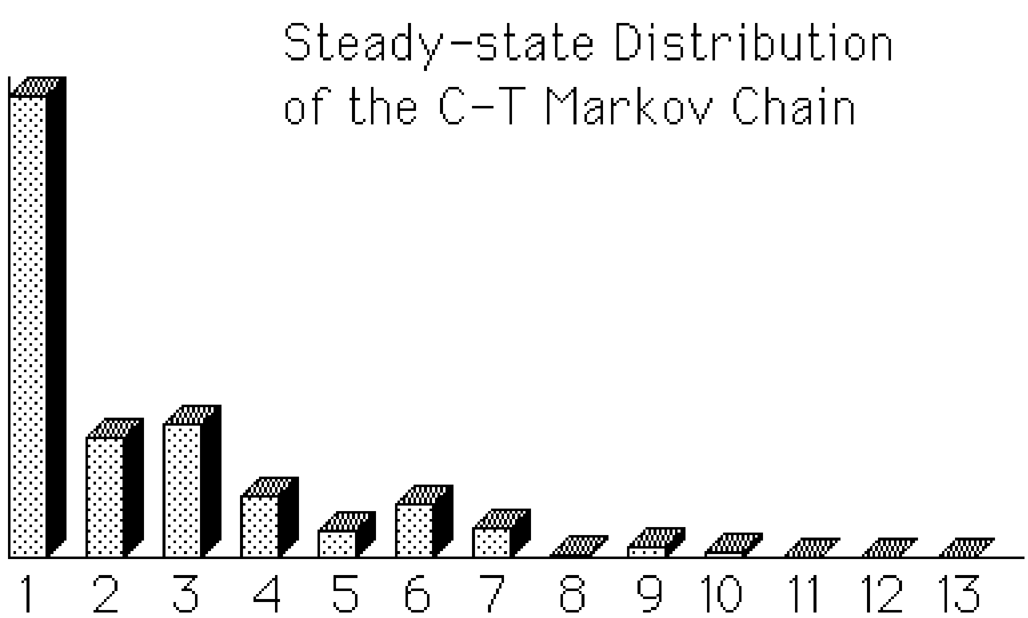
Steady-State Distribution

i	j	t	PI
1	0	0	0.501827
2	1	1	0.132482
3	1	2	0.147203
4	1	3	0.066910
5	2	1	0.029394
6	2	2	0.060558
7	2	3	0.034660
8	3	1	0.003982
9	3	2	0.012547
10	3	3	0.008307
11	4	1	0.000199
12	4	2	0.001102
13	4	3	0.000828

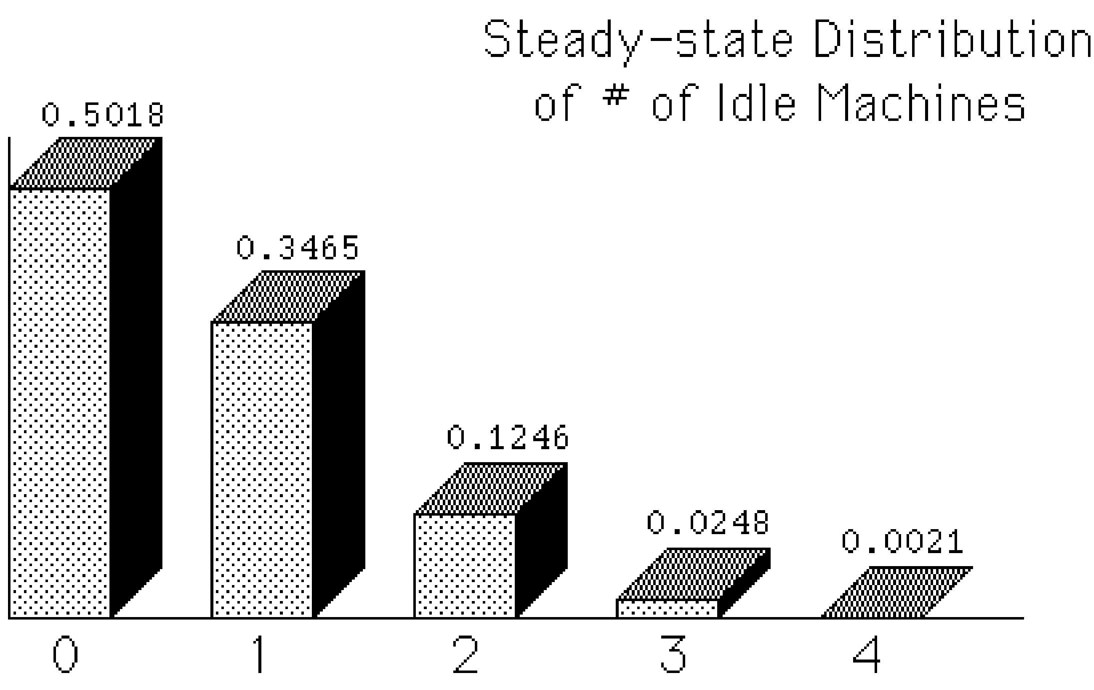


Distribution of # jobs in the System

j	P
0	0.501827
1	0.346595
2	0.124612
3	0.024837
4	0.002129

Mean number of jobs in system = 0.6788461127

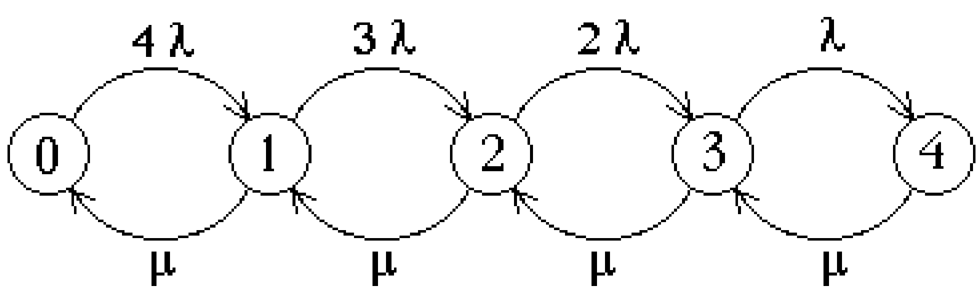
That is, an average of 0.6788 machines are idle at any time, a utilization of 83.03%



Suppose that we use the M/M/1/N/N "approximation" to this problem.

The variance of the service time of the M/M/1/N/N system is larger.

Birth-Death Model



M/M/1/N/N

with $\lambda = 1/5$ per minute

$\mu = \frac{4}{3}$ per minute

Steady-State Distribution

i	Pi
0	0.509385
1	0.305631
2	0.137534
3	0.041260
4	0.006189

The mean number of customers in the system
(including the one being served) is: 0.7292361766

The average arrival rate of customers is 0.6541527647

Using Little's formula, the average time
spent in the system, per customer, is W = 1.114779629

$$\text{Expected utilization} = \frac{4-0.729236}{4} = 81.77\%$$

Suppose that expected processing time is 3 minutes, rather than 5 minutes,
i.e., $\lambda = 1/3$
using the original model, i.e., not the birth-death model

Distribution of # jobs in the System

j	P
0	0.291087
1	0.366554
2	0.242240
3	0.087166
4	0.012954

Mean number of jobs in system = 1.16434698

$$\text{Expected utilization} = \frac{4-1.1643}{4} = 70.89\%$$

M/M/1/N/N

Steady-State Distribution

with $\lambda = 1/3$ per minute

$\mu = \frac{4}{3}$ per minute

i	Pi
0	0.310680
1	0.310680
2	0.233010
3	0.116505
4	0.029126

The mean number of customers in the system (including the one being served) is: 1.242718447

The average arrival rate of customers is 0.9190938511

Using Little's formula, the average time spent in the system, per customer, is W = 1.352112676

$$\text{Expected utilization} = \frac{4-1.242718}{4} = 68.93\%$$

Suppose that expected processing time is 1 minute, i.e.,

$\lambda = 1/10$

Distribution of # jobs in the System

j	P
0	0.724170
1	0.233303
2	0.038766
3	0.003612
4	0.000149

Mean number of jobs in system = 0.3222666048

$$\text{Expected utilization} = \frac{4-0.32227}{4} = 91.93\ \%$$

M/M/1/N/N

with $\lambda = \frac{1}{10}$ per minute

$\mu = \frac{4}{3}$ per minute

Steady-State Distribution

i	Pi
0	0.725487
1	0.217646
2	0.048970
3	0.007346
4	0.000551

The mean number of customers in the system
(including the one being served) is: 0.3398271981

The average arrival rate of customers is 0.3660172802

Using Little's formula, the average time
spent in the system, per customer, is W = 0.9284457769

$$\text{Expected utilization} = \frac{4-0.339827}{4} = 91.5\%$$

Summary: Expected Utilization, using the 2 models

λ	M/M/1/N/N	$M/E_3/1/N/N$
$\frac{1}{3}$	68.93%	70.89%
$\frac{1}{5}$	81.77%	83.03%
$\frac{1}{10}$	91.5%	91.93 %

Assuming exponential dist'n for service time, i.e., a larger variance, leads to an underestimate of the utilization!

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