Consider the inequality-constrained problem:

\[
\begin{align*}
\text{Minimize } & f(x) \\
\text{subject to } & g_i(x) \leq 0, \ i = 1, 2, \ldots, m \\
& x \in X
\end{align*}
\]

Define the Lagrangian function:

\[
L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)
\]
Based upon this Lagrangian function, we define two functions:

\[ L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \]

Primal Objective

\[ \overline{L}(x) \equiv \max_{\lambda \geq 0} L(x, \lambda) \]

Fix "x" and maximize with respect to the Lagrange multiplier

Dual Objective

\[ \tilde{L}(\lambda) \equiv \min_{x \in X} L(x, \lambda) \]

Fix the Lagrange multiplier and minimize w.r.t. "x"

Weak Duality Relationship: for all \( x \in X \) and \( \lambda \geq 0 \),

\[
\max_{\lambda \geq 0} L(x, \lambda) \equiv \overline{L}(x) \geq L(x, \lambda) \geq \tilde{L}(\lambda) \equiv \min_{x \in X} L(x, \lambda)
\]
\[ L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \]

**Primal Objective**

\[ \overline{L}(x) \equiv \max_{\lambda \geq 0} L(x, \lambda) \]

\[ = \begin{cases} 
  f(x) & \text{if } g_i(x) \leq 0 \ \forall \ i \\
  +\infty & \text{if } g_i(x) > 0 \ \text{for some } i
\end{cases} \]

- If \( g_i(x) \leq 0 \ \forall \ i \) then optimal \( \lambda_i \)'s are zero;
- otherwise, if \( g_i(x) > 0 \)
  - for some \( i \), \( L(x, \lambda) \) is unbounded above as \( \lambda_i \to +\infty \)

\[ L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \]

**Primal Problem**

Minimize \( \overline{L}(x) \) \( \quad \) where \( \overline{L}(x) \equiv \max_{\lambda \geq 0} L(x, \lambda) \)

**Dual Problem**

Maximize \( \hat{L}(\lambda) \) \( \quad \) where \( \hat{L}(\lambda) \equiv \min_{x \in X} L(x, \lambda) \)
Primal Problem

Minimize $\bar{L}(x)$

where $\bar{L}(x) = \begin{cases} 
  f(x) & \text{if } g_i(x) \leq 0 \ \forall \ i \\
  +\infty & \text{if } g_i(x) > 0 \text{ for some } i 
\end{cases}$

If there exists an $x$ feasible in $\{g_i(x) \leq 0 \ \forall \ i\}$, then we can restrict our search for the minimizing $x$ to such $x$'s, and therefore

$$\underset{x \in X}{\text{Minimum}} \bar{L}(x) = \underset{x \in X}{\text{Minimum}} \{ f(x) \mid g_i(x) \leq 0 \ \forall \ i \} $$

And so we see that

Primal Problem

Minimize $\bar{L}(x)$

is equivalent to our original problem:

Minimize $f(x)$

subject to

$g_i(x) \leq 0, \ i = 1, 2, \cdots m$

$x \in X$
Weak Duality Relationship

For all $x \in \mathcal{X}$ and $\lambda \geq 0$,

\[ \bar{L}(x) \geq \bar{L}(x, \lambda) \geq \hat{L}(\lambda) \]

Primal objective \geq Dual objective

In particular, if $x^*$ and $\lambda^*$ are the primal and dual optima, respectively, then

\[ \bar{L}(x^*) \geq \hat{L}(\lambda^*) \]

i.e.,

\[ \bar{L}(x^*) - \hat{L}(\lambda^*) \geq 0 \]

Duality Gap

That is, any feasible dual solution gives a lower bound on all primal solutions, including of course the optimal.... this property is often used to advantage in branch-and-bound algorithms for combinatorial problems.
**Definition**

$(\bar{x}, \bar{\lambda})$ is a *saddlepoint* of $L(x, \lambda)$ if

$$L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \quad \forall \ x \in X$$

(which implies that $L(\bar{x}) = L(\bar{x}, \bar{\lambda})$)

and

$$L(\bar{x}, \bar{\lambda}) \geq L(\bar{x}, \lambda) \quad \forall \ \lambda \geq 0$$

(which implies that $\lambda(\bar{\lambda}) = L(\bar{x}, \bar{\lambda})$)

If $(\bar{x}, \bar{\lambda})$ is a saddlepoint of $L(x, \lambda)$

then

$$\bar{L}(\bar{x}) = L(\bar{x}, \bar{\lambda}) = \bar{L}(\bar{\lambda})$$

*primal* objective

*dual* objective

so that the duality gap is zero!
EXAMPLE

Minimize \(4x_1^2 + 2x_1x_2 + x_2^2\)
subject to \(3x_1 + x_2 \geq 6\)
\(x_1 \geq 0, x_2 \geq 0\)

Define: \(g(x) = 6 - 3x_1 - x_2\)
\(X = \{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0\}\)

The Lagrangian is
\[L(x, \lambda) = 4x_1^2 + 2x_1x_2 + x_2^2 + \lambda(6 - 3x_1 - x_2)\]

Dual objective:
\[\tilde{L}(\lambda) = \min_{x \geq 0} \{4x_1^2 + 2x_1x_2 + x_2^2 + \lambda(6 - 3x_1 - x_2)\}\]

The K–K–T necessary conditions for optimality of \(x_1, x_2 \geq 0\) are:

(for \(\lambda\) fixed)

\[\frac{\partial L}{\partial x_1} = 8x_1 + 2x_2 - 3\lambda \geq 0\]
\[\frac{\partial L}{\partial x_2} = 2x_1 + 2x_2 - \lambda \geq 0\]
\[x_1 \left[\frac{\partial L}{\partial x_1}\right] = 0, \quad x_2 \left[\frac{\partial L}{\partial x_2}\right] = 0\]

with solution:
\[x_1^*(\lambda) = \frac{\lambda}{3}, \quad x_2^*(\lambda) = \frac{\lambda}{6}\]
\(x_1, x_2 \geq 0 \forall  \lambda \geq 0\)

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And so the dual objective is

\[ \tilde{L}(\lambda) = L(\lambda_3, \lambda_6, \lambda) = 6\lambda - \frac{7}{12}\lambda^2 \leftarrow \text{a CONCAVE function of } \lambda \]

and the dual problem is

\[
\text{Maximize } 6\lambda - \frac{7}{12}\lambda^2 \\
\text{subject to } \lambda \geq 0
\]
Dual problem:

Maximize: \[ 6 \lambda - \frac{7}{12} \lambda^2 \]
subject to: \[ \lambda \geq 0 \]

The necessary (and sufficient) conditions for optimality are

\[
\frac{dL(\lambda)}{d\lambda} = 6 - 2 \left( \frac{7}{12} \right) \lambda \leq 0, \quad \lambda \left[ \frac{dL(\lambda)}{d\lambda} \right] = 0
\]

\[ \Rightarrow \quad \lambda^* = \frac{36}{7} \quad \widehat{L}(\lambda^*) = \widehat{L} \left( \frac{36}{7} \right) = \frac{108}{7} \]

The corresponding values of \( x^* \) which optimize the Lagrangian subproblem, i.e., the problem of evaluating the dual objective \( \widehat{L} \), are:

\[
\begin{align*}
\left\{ \begin{array}{l}
x_1^*(\lambda^*) = \frac{\lambda^*}{3} = \frac{36/7}{3} = \frac{12}{7}, \\
x_2^*(\lambda^*) = \frac{\lambda^*}{6} = \frac{36/7}{6} = \frac{6}{7}
\end{array} \right.
\]

at which the primal objective, \( 4x_1^2 + 2x_1x_2 + x_2^2 \), also has the value \( \frac{108}{7} \)
\( L(x) = \begin{cases} 
4x_1^2 + 2x_1x_2 + x_2^2 & \text{if } 3x_1 + x_2 \leq 6, \ x \geq 0 \\
\infty & \text{otherwise}
\end{cases} \)

\[ x_1^* = \frac{12}{7}, \ x_2^* = \frac{6}{7}, \ \bar{L}(x^*) = \frac{108}{7} \]

\[ \hat{L}(\lambda) = 6\lambda - \frac{7}{12}\lambda^2, \ \lambda \geq 0 \]

\[ \lambda^* = \frac{36}{7}, \ \bar{L}(\lambda^*) = \frac{108}{7} \]

\[ \bar{L}(x^*) = \hat{L}(\lambda^*) \]

No Duality Gap!

**Geometric Interpretation**

Define \( G = \{ (z_1, z_2) \mid z_1 = g(x), \ z_2 = f(x) \text{ for } x \in X \} \)

Primal can be restated as:

Minimize \( f(x) \)

subject to \( g(x) \leq 0 \)

\( x \in X \)
For fixed $\lambda$, 
\[
\widetilde{L}(\lambda) = \min_{z \in G} \{ z_2 + \lambda z_1 \}
\]

$\widetilde{L}(\lambda)$ is the $z_2$-intercept of the supporting hyperplane of $G$ with slope $-\lambda$.

Dual problem is to find support with maximum $z_2$-intercept.

The optimal $\lambda$ is that for which the $z$-intercept of the supporting hyperplane (in this case, line) is maximized.

Slope is $-\lambda^*$.
When $G$ is nonconvex, a duality gap is possible!

**EXAMPLE**

integer linear program

Minimize $3x_1 + 7x_2 + 10x_3$
subject to $x_1 + 3x_2 + 5x_3 \geq 7$
$x_j \in \{0,1\}$, $j=1,2,3$

Define:

$$X = \{ x = (x_1,x_2,x_3) \mid x_j \in \{0,1\} \}$$

$= \{0,1\} \times \{0,1\} \times \{0,1\}$ Cartesian product

$g(x) = 7 - x_1 - 3x_2 - 5x_3$

Lagrangian function:

$$L(x,\lambda) = 3x_1 + 7x_2 + 10x_3 + \lambda(7 - x_1 - 3x_2 - 5x_3)$$

$$= (3 - \lambda)x_1 + (10 - \lambda)x_2 + (5 - \lambda)x_3 + 7\lambda$$

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Dual objective: \[ \tilde{L}(\lambda) \equiv \min_{x_j \in \{0,1\},j=1,2,3} L(x,\lambda) \]

\[ \tilde{L}(\lambda) = \min_{x_j \in \{0,1\}} (3 - \lambda)x_1 + (10 - 3\lambda)x_2 + (5 - 5\lambda)x_3 + 7\lambda \]

Given a value of \( \lambda \), the optimal \( x_j^*(\lambda) \) is 0 if its coefficient is positive, and 1 otherwise.

For example, if \( \lambda = 2.5 \),

\[ L(x,2.5) = 0.5x_1 - 0.5x_2 - 2.5x_3 + 17.5 \]

\( x_1^*(2.5) = x_2^*(2.5) = 0 \), \( x_3^*(2.5) = 1 \)

\[ \tilde{L}(2.5) = 14.5 \]

Thus,

\[ x_1^*(\lambda) = \begin{cases} 1 & \text{if } 3 - \lambda \leq 0, \text{ i.e., } \lambda \geq 3 \\ 0 & \text{otherwise} \end{cases} \]

\[ x_2^*(\lambda) = \begin{cases} 1 & \text{if } 7 - 3\lambda \leq 0, \text{ i.e., } \lambda \geq 7/3 \\ 0 & \text{otherwise} \end{cases} \]

\[ x_3^*(\lambda) = \begin{cases} 1 & \text{if } 10 - 5\lambda \leq 0, \text{ i.e., } \lambda \geq 2 \\ 0 & \text{otherwise} \end{cases} \]

will minimize \( L(x,\lambda) \) for a given \( \lambda \)
When the coefficient of $x_j$ is zero, then both 0 & 1 are optimal values for that variable.
By inspection of the graph of $\widehat{L}(\lambda)$, we see that the optimal dual solution is

$$\lambda^* = 7/3 \quad \Rightarrow \quad \widehat{L}(\lambda^*) = 44/3$$

At $\lambda^*$, both $x'=(0,0,1)$ and $x''=(0,1,1)$ minimize $L(x,\lambda)$.

But $x'$ is infeasible in $x_1 + 3x_2 + 5x_3 \geq 7$

and $x''$ violates the complementary slackness condition:

$$\lambda^* \left[ 7 - x''_1 - 3x''_2 - 5x''_3 \right] = 0$$

$$\frac{7}{3} \left[ -1 \right]$$

Neither $x'$ nor $x''$ are optimal in the primal problem!

---

Solving the primal problem by complete enumeration:

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$g(x)$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>17</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>13</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>20</td>
</tr>
</tbody>
</table>
Primal solution
\[ \bar{L}(x^*) = 17 = 51/3 \]

Dual solution
\[ \widehat{L}(\lambda^*) = 44/3 \]

Duality Gap > 0!
\[ \bar{L}(x^*) - \widehat{L}(\lambda^*) = 7/3 \]
Consider the problem:

Minimize $f(x)$
subject to
$g_i(x) \leq 0, \ i=1,2,\cdots m$
$x \in X$

where $f(x)$ & $g_i(x)$ are convex functions, and
$x$ is a convex set.

Let $\bar{\lambda} \geq 0$ and $\bar{x} \in X$....

Then $(\bar{x}, \bar{\lambda})$ is a saddlepoint of the Lagrangian function $L(x, \lambda)$ if & only if

\[
\begin{align*}
\bullet & \quad \bar{x} \text{ minimizes } L(x, \lambda) = f(x) + \bar{\lambda}^T g(x) \text{ over } X \\
\bullet & \quad g_i(\bar{x}) \leq 0 \quad \text{for each } \ i=1,2,\cdots m \\
\bullet & \quad \bar{\lambda}_i g_i(\bar{x}) = 0 \quad \text{which implies } f(\bar{x}) = L(\bar{x}, \bar{\lambda})
\end{align*}
\]

(If a saddlepoint exists, then the duality gap is zero!)
If \((\bar{x}, \bar{\lambda})\) is a saddlepoint for \(L(x, \lambda)\)

then \(\bar{x}\) solves the primal problem:

\[
\begin{align*}
\text{Minimize } & f(x) \\
\text{subject to } & g_i(x) \leq 0, \; i = 1,2,\ldots,m \\
& x \in X
\end{align*}
\]

and \(\bar{\lambda}\) solves the dual problem:

\[
\begin{align*}
\text{Maximize } & \hat{L}(\lambda) \\
\text{subject to } & \lambda \geq 0
\end{align*}
\]

where \(\hat{L}(\lambda) = \min_{x \in X} L(x, \lambda)\)

---

**STRONG DUALITY THEOREM**

Consider the primal problem: Find

\[
\Phi = \inf f(x)
\]

subject to

\[
\begin{align*}
g_i(x) & \leq 0, \; i = 1,2,\ldots,m_1 \\
h_i(x) &= 0, \; i = 1,2,\ldots,m_2 \\
x & \in X
\end{align*}
\]

where

\[
X \subseteq \mathbb{R}^n \text{ is nonempty & convex} \\
f(x) & \text{& } g_i(x) \text{ are convex} \\
h_i(x) & \text{ are linear}
\]

("\(\infimum\) may be replaced by "minimum" if the minimum is achieved at some \(x\).")

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Define the Dual Problem:

Find

\[ \Psi = \sup_{\lambda \geq 0} \tilde{L}(\lambda, \mu) \]

where

\[ \tilde{L}(\lambda, \mu) = \inf_{x \in X} \{ f(x) + \lambda^T g(x) + \mu^T h(x) \} \]

Assume also that the following "Constraint Qualification" holds:

There exists \( \hat{x} \) such that

\[ g_i(\hat{x}) < 0, \quad i = 1, 2, \ldots, m_1 \]

\[ h_i(\hat{x}) = 0, \quad i = 1, 2, \ldots, m_2 \]

& \( 0 \in \text{int} \, h(X) \)
Then \( \Phi = \Psi \)

i.e., there is no duality gap!

Furthermore, if \( \Phi > -\infty \) then

- \( \Psi = \hat{L}(\lambda^*, \mu^*) \) for some \( \lambda^* \geq 0 \)
- if \( x^* \) solves the primal, it satisfies complementary slackness, i.e.,
  \[ \lambda_i^* g_i(x^*) = 0 \quad \forall \ i \]

### Example

Minimize \( f(x) = -x^2 - x^3 \)

subject to \( x^2 \leq 1 \)

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?
Graphically, we can see that \( x^* = 1, \ f(x^*) = -2 \)

\[
f(x) = -x^2 - x^3
\]

Lagrangian function

\[
L(x, \lambda) = -x^2 - x^3 + \lambda (x^2 - 1)
\]

\[
\frac{dL}{dx} = -2x - 3x^2 + 2\lambda \quad x = 0
\]

\[
x^2 \leq 1
\]

\[
\lambda (x^2 - 1) = 0
\]

\[
\lambda \geq 0
\]

KKT points are

\[
(x, \lambda) = (-\frac{2}{3}, 0) \quad (0, 0) \quad (1, \frac{5}{2})
\]

\[
L(x, \lambda) = -\frac{4}{27} \quad 0 \quad -2
\]
**Dual Problem**

Maximize \( \hat{L}(\lambda) \)
subject to \( \lambda \geq 0 \)

where \( \hat{L}(\lambda) \equiv \min_{x \in X} L(x, \lambda) \)

\[
= \min_{x \in X} \left\{ -x^2 - x^3 + \lambda (x^2 - 1) \right\}
\]

\[
= -\infty \quad \text{for all } \lambda \geq 0
\]

\[\implies \text{Maximum } \hat{L}(\lambda) = -\infty\]

\[G = \left\{ (z_1, z_2) \mid z_1 = g(x), z_2 = f(x) \text{ for some } x \right\}\]

\[z_2 = f(x) = -x^2 - x^3\]

\[z_1 = g(x) = x^2 - 1 \quad \Rightarrow \quad x = \pm \left(1 + z_1\right)^{1/2}\]

\[\implies G = \left\{ (z_1, z_2) \mid z_2 = -(1 + z_1) \pm \left(1 + z_1\right)^{3/2}\right\}\]
The set $G$ consists of the curve below:

There is no nonvertical support of $G$ which has negative (= $-\lambda$) slope!

**EXAMPLE**

Minimize $-(x - 4)^2$

subject to $1 \leq x \leq 6$

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?
EXAMPLE

Minimize \( f(x,y) = x \) subject to \( g(x,y) = x^2 + y^2 \leq 1 \)

• Write the Lagrangian function
• State the KKT optimality conditions
• Solve graphically, and verify that the KKT conditions are satisfied at the optimum
• State the Lagrangian dual objective
• Solve the dual problem
• Is there a duality gap?

EXAMPLE

Minimize \( (x - 4)^2 \) subject to \( 1 \leq x \leq 3 \)

• Write the Lagrangian function
• State the KKT optimality conditions
• Solve graphically, and verify that the KKT conditions are satisfied at the optimum
• State the Lagrangian dual objective
• Solve the dual problem
• Is there a duality gap?