Linear Programming Models

Definitions

linear function

$$f(x_1, x_2, \cdots x_n) = \sum_{j=1}^{n} c_j x_j$$

$$= c_1 x_1 + c_2 x_2 + \cdots c_n x_n$$

linear inequality

$$\sum_{j=1}^{n} a_j x_j \leq b$$

Linear Programming

--an optimization problem for which:

- we maximize or minimize a linear function of the *decision variables* (the *objective* function)
- the values of the decision variables must satisfy a set of *constraints*, each consisting of a linear equation or a linear inequality
- a sign restriction (nonnegative, i.e.≥0, nonpositive, i.e., ≤0) may be associated with each decision variable.

Assumptions of LP Model

- **Proportionality:** the contribution of each variable to the objective function or constraint function is proportional to the value of that variable

E.g., doubling a decision variable will double the contribution to the cost function.

This implies that there are no economies of scale!

Assumptions of LP Model

- **Additivity:** The contribution of any variable to the objective or a constraint function is independent of the values of the other decision variables.
- **Certainty:** All parameters (cost coefficients, right-hand-sides of constraints, etc.) are known with certainty, i.e., are not random.
- **Continuity:** Each decision variable can assume real values, i.e., are not restricted to a discrete set of values, e.g., integers

Examples

- **Investment Planning**
- **Feedem-Speedem Airlines**
- **Fine-Webb Paper Company**

Investment Planning

An investor has money-making activities *A* and *B* available at the beginning of each of the next 5 years, while activities *C* and *D* are available only once.

⏎

- Each $1 invested in *A* at the beginning of one year returns $1.40 (a profit of 40¢) two years later
- Each $1 invested in *B* at the beginning of one year returns $1.70 three years later.
- Each $1 invested in *C* at the beginning of the *second* year returns $2 four years later.
- Each $1 invested in *D* at the beginning of the *fifth* year returns $1.30 one year later.

The investor begins with $10,000.

He wishes to know which investments to make so as to maximize his accumulated wealth at the beginning of the sixth year.

Definition of Decision Variables

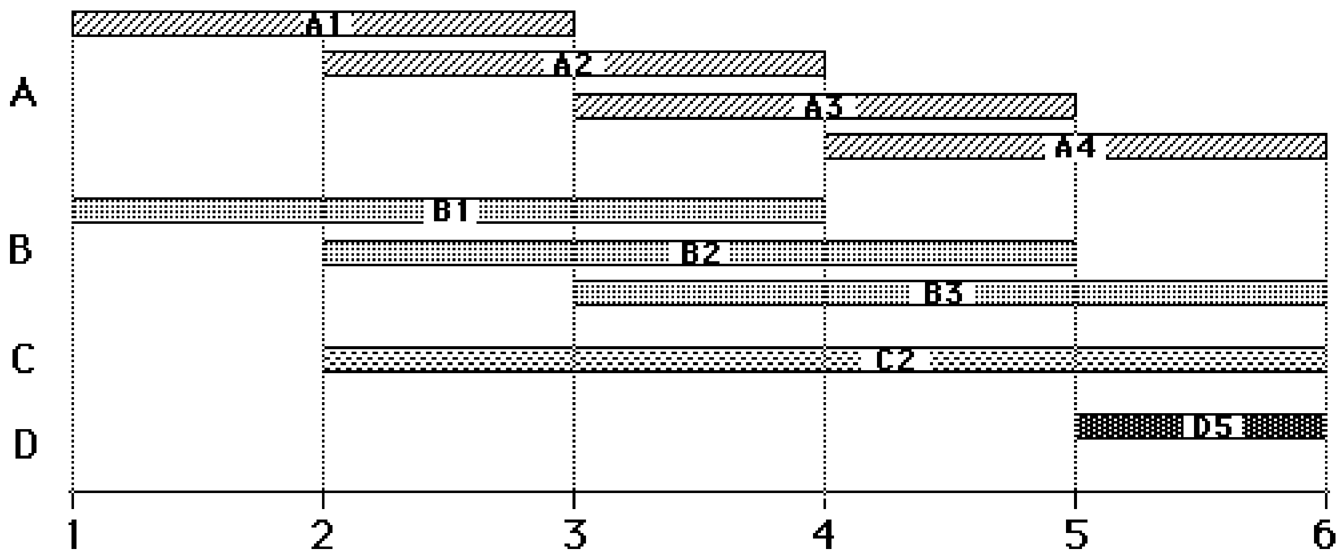
At = $ invested in *A* at the beginning of year *t*, t=1,2,3,4

Bt = $ invested in *B* at the beginning of year *t*, t=1,2,3

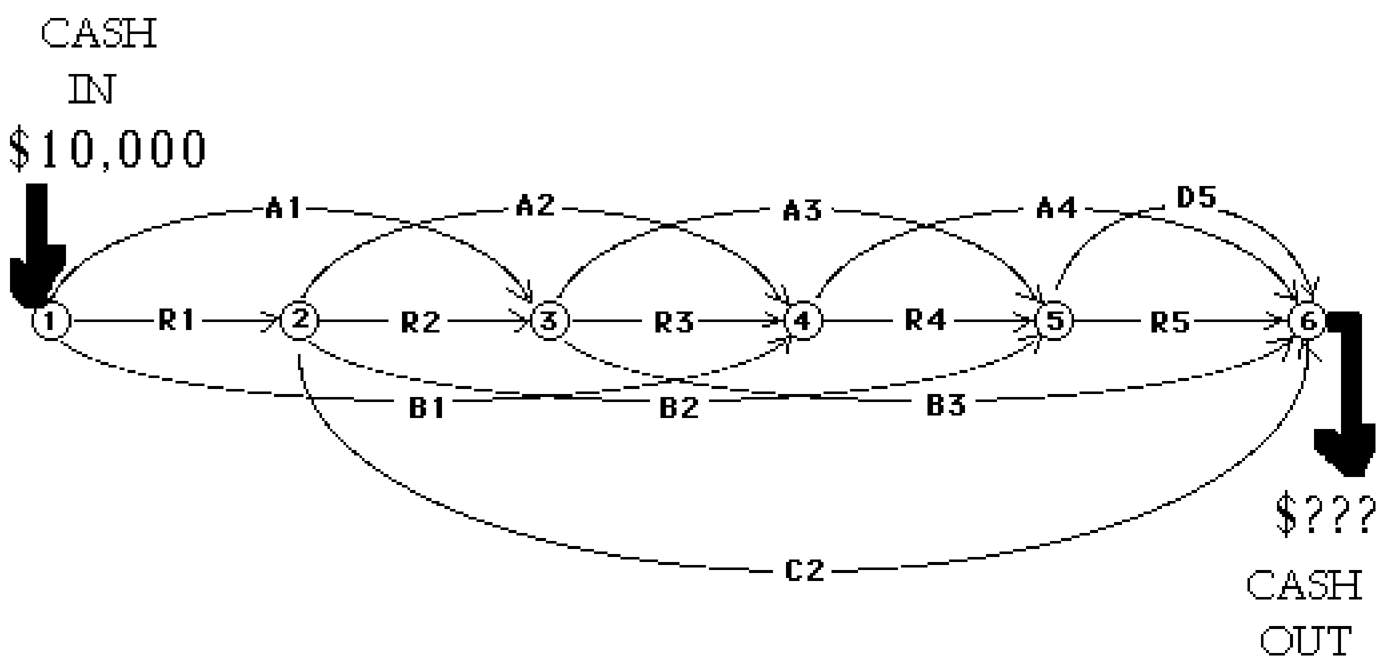
C2 = $ invested in *C* at the beginning of year 2

D5 = $ invested in *D* at the beginning of year 5

Rt = $ held in Reserve from year t until year t+1, t=1,2,3,4



©D.L.Bricker, U. of Iowa, 1998



"Constraints" for each year (1 through 5)

Amount Invested	=	**Amount Available**
A1 + B1 + R1	=	10000
A2 + B2 + C2 + R2	=	R1
A3 + B3 + R3	=	R2 + 1.4 A1
A4 + R4	=	R3 + 1.4A2 + 1.7B1
D5 + R5	=	R4 + 1.4A3 + 1.7B2
Accumulated wealth	=	R5 + 1.4A4 + 1.7B3 + 2C2 + 1.3D5

to be maximized

Accumulated wealth

```
MAX    2 C2 + 1.7 B3 + 1.4 A4 + 1.3 D5 + R5
SUBJECT TO
    2)  A1 + B1 + R1                     =  10000
    3)  C2 - R1 + A2 + B2 + R2           =  0
    4)  B3 - 1.4 A1 - R2 + A3 + R3       =  0
    5)  A4 - 1.7 B1 - 1.4 A2 - R3 + R4   =  0
    6)  D5 + R5 - 1.7 B2 - 1.4 A3 - R4   =  0
END

OBJECTIVE FUNCTION VALUE

  1)    25480.0000
```

note that all variables have been transferred to left-hand-side

VARIABLE	VALUE	REDUCED COST
C2	.000000	.210000
B3	.000000	.120000
A4	.000000	.000000
D5	19600.000000	.000000
R5	.000000	.300000
A1	10000.000000	.000000
B1	.000000	.168000
R1	.000000	.338000
A2	.000000	.250000
B2	.000000	.000000
R2	.000000	.390000
A3	14000.000000	.000000
R3	.000000	.420000
R4	.000000	.100000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	2.548000
3)	.000000	2.210000
4)	.000000	1.820000
5)	.000000	1.400000
6)	.000000	1.300000

Summary:

VARIABLE	VALUE	REDUCED COST
A1	10000.000000	.000000
A3	14000.000000	.000000
D5	19600.000000	.000000

OBJECTIVE FUNCTION VALUE 25480.0000

That is, invest all $10K in A at beginning of year 1.

This will grow in value to $14K at beginning of year 3 (2 years later). Re-invest all of this in A again.

Investment A will grow in value to $19.6K at the beginning of year 5.

Invest all of this in D for the year beginning of year 5; this will grow in value to $25.480K at the beginning of year 6.

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
C2	2.000000	.210000	INFINITY
B3	1.700000	.120000	INFINITY
A4	1.400000	.098824	.100000
D5	1.300000	.100000	.085714
R5	1.000000	.300000	INFINITY
A1	.000000	INFINITY	.168000
B1	.000000	.168000	INFINITY
R1	.000000	.338000	INFINITY
A2	.000000	.250000	INFINITY
B2	.000000	.338000	.210000
R2	.000000	.390000	INFINITY

A3	.000000	.390000	.120000
R3	.000000	.420000	INFINITY
R4	.000000	.100000	INFINITY

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	10000.000000	INFINITY	10000.000000
3	.000000	INFINITY	.000000
4	.000000	INFINITY	14000.000000
5	.000000	INFINITY	.000000
6	.000000	INFINITY	19600.000000

THE TABLEAU

ROW	(BASIS)	C2	B3	A4	D5	R5	A1	B1
1	ART	.210	.120	.0	.0	.300	.0	.168
2	A1	.000	.000	.0	.0	.000	1.0	1.000
3	B2	1.000	.000	.0	.0	.000	.0	.000
4	A3	.000	1.000	.0	.0	.000	.0	1.400
5	A4	.000	.000	1.0	.0	.000	.0	-1.700
6	D5	1.700	1.400	.0	1.0	1.000	.0	1.960

ROW	R1	A2	B2	R2	A3	R3	R4	*rhs*
1	.338	.250	.0	.390	.0	.420	.1	25480.0
2	1.000	.000	.0	.000	.0	.000	.0	10000.0
3	-1.000	1.000	1.0	1.000	.0	.000	.0	.0
4	1.400	.000	.0	-1.000	1.0	1.000	.0	14000.0
5	.000	-1.400	.0	.000	.0	-1.000	1.0	.0
6	.260	1.700	.0	.300	.0	1.400	-1.0	19600.0

Feedem-Speedem Airline Company

must decide how many new stewardesses to hire and train over the next six months.

Requirements (stewardess-flight-hours):

Month	1 Jan.	2 Feb.	3 Mar.	4 Apr.	5 May	6 June
Rqmt.	8000	9000	7000	10000	9000	11000

- One month of training is required before a new stewardess can be put on a regular flight.
- A trainee requires 100 hours of supervision by an experienced stewardess during this month. (While supervising a trainee, the stewardess is not available for flight service.)
- An experienced stewardess can work up to 150 hours/month.

- The company has 60 experienced stewardesses available at the beginning of January.
- Approximately 10% of the experienced stewardesses quit their jobs each month.
- Costs (salaries & benefits) are
  - \$1800/month for trainees
  - \$3400/month for experienced stewardess

Find a plan for hiring which will minimize the total costs.

Definition of Decision Variables

Y_t = # of persons hired at beginning of month t, t=1,2,3,4,5,6

X_t = # of trained persons available for flight in month t, t=1,2,3,4,5,6

```
MIN 1800 Y1 + 3400 X1 + 1800 Y2 + 3400 X2 + 1800 Y3 + 3400 X3
  + 1800 Y4 + 3400 X4 + 1800 Y5 + 3400 X5 + 1800 Y6 + 3400 X6
SUBJECT TO
   2) - 100 Y1 + 150 X1 >=  8000
   3) - 100 Y2 + 150 X2 >=  9000
   4) - 100 Y3 + 150 X3 >=  7000
   5) - 100 Y4 + 150 X4 >=  10000
   6) - 100 Y5 + 150 X5 >=  9000
   7) - 100 Y6 + 150 X6 >=  11000
   8) - Y1 - 0.9 X1 + X2 <=  0
   9) - Y2 - 0.9 X2 + X3 <=  0
  10) - Y3 - 0.9 X3 + X4 <=  0
  11) - Y4 - 0.9 X4 + X5 <=  0
  12) - Y5 - 0.9 X5 + X6 <=  0
  13)  X1 <=  60
END
```

flight-hour reqmts. (constraints 2–7)

inequality assumes that airline may lay off workers (constraints 8–12)

Optimal Solution:

OBJECTIVE FUNCTION VALUE

1) 1410977.00

CONTINUOUS LP MODEL

VARIABLE	VALUE
Y1	8.630985
X1	59.087322
Y2	2.714363
X2	61.809574
Y3	17.514470
X3	58.342980
Y4	5.034724
X4	70.023150
Y5	12.083333
X5	68.055560
Y6	.000000
X6	73.333330

note that the optimal values of these variables are NOT integer!

- ☞ Reduced costs
- ☞ Dual variables
- ☞ Objective coefficient ranges
- ☞ Right-hand-side ranges
- ☞ Tableau
- ☞ Integer solution

VARIABLE	VALUE	REDUCED COST
Y1	8.630985	.000000
X1	59.087322	.000000
Y2	2.714363	.000000
X2	61.809574	.000000
Y3	17.514470	.000000
X3	58.342980	.000000
Y4	5.034724	.000000
X4	70.023150	.000000
Y5	12.083333	.000000
X5	68.055560	.000000
Y6	.000000	6934.948200
X6	73.333330	.000000

©D.L.Bricker, U. of Iowa, 1998

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	-7.416667
3)	.000000	-18.006944
4)	.000000	-22.419563
5)	.000000	-24.258150
6)	.000000	-25.024230
7)	.000000	-51.349483
8)	.000000	2541.667000
9)	.000000	3600.695000
10)	.000000	4041.956000
11)	.000000	4225.815000
12)	.000000	4302.423000
13)	.912676	.000000

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
Y1	1800.000000	1977.778000	4066.667000
X1	3400.000000	INFINITY	1780.000000
Y2	1800.000000	4801.852000	5761.112000
X2	3400.000000	INFINITY	4321.667000
Y3	1800.000000	5978.550200	6467.129300
X3	3400.000000	INFINITY	5380.695300
Y4	1800.000000	6468.840000	6761.304000
X4	3400.000000	INFINITY	5821.956000
Y5	1800.000000	6673.128000	6883.876400
X5	3400.000000	INFINITY	6005.815400
Y6	1800.000000	INFINITY	6934.948200
X6	3400.000000	INFINITY	7702.423000

©D.L.Bricker, U. of Iowa, 1998

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	8000.000000	219.042320	14180.960000
3	9000.000000	525.701530	3314.298300
4	7000.000000	1261.684000	1042.315400
5	10000.000000	1342.593000	2501.557000
6	9000.000000	3222.222400	1933.334000
7	11000.000000	10900.950000	2900.000200
8	.000000	13.809580	2.190423
9	.000000	4.342981	5.257016
10	.000000	10.423152	12.616840
11	.000000	8.055558	30.280410
12	.000000	19.333333	72.672981
13	60.000000	INFINITY	.912676

THE TABLEAU

ROW	(BASIS)	Y1	X1	Y2	X2	Y3	X3
1	ART	0.0	.0	0.0	0.0	0.0	.0
2	X1	0.0	1.0	0.0	0.0	0.0	.0
3	X2	.0	.0	0.0	1.0	0.0	0.0
4	X3	.0	.0	.0	.0	0.0	1.0
5	X4	.0	.0	.0	.0	.0	.0
6	X5	.0	.0	.0	.0	.0	.0
7	X6	.0	.0	.0	.0	.0	.0
8	Y1	1.0	.0	.0	0.0	.0	0.0
9	Y2	.0	.0	1.0	.0	.0	0.0
10	Y3	.0	.0	.0	.0	1.0	.0
11	Y4	.0	.0	.0	.0	.0	.0
12	Y5	.0	.0	.0	.0	.0	.0
13	SLK13	0.0	0.0	0.0	0.0	0.0	.0

©D.L.Bricker, U. of Iowa, 1998

THE TABLEAU

ROW	Y4	X4	Y5	X5	Y6	X6	SLK 2
1	.0	0.0	.0	0.0	6934.948	0.0	7.417
2	.0	0.0	.0	0.0	-.008	0.0	-.004
3	0.0	.0	.0	0.0	-.020	.0	.000
4	0.0	0.0	0.0	.0	-.048	0.0	.000
5	0.0	1.0	0.0	0.0	-.116	0.0	.000
6	.0	.0	0.0	1.0	-.278	0.0	.000
7	.0	.0	.0	.0	-.667	1.0	.000
8	0.0	.0	.0	0.0	-.013	.0	.004
9	.0	0.0	0.0	.0	-.030	.0	.000
10	.0	0.0	.0	0.0	-.072	.0	.000
11	1.0	.0	.0	0.0	-.174	.0	.000
12	.0	.0	1.0	.0	-.417	0.0	.000
13	.0	0.0	.0	0.0	.008	0.0	.004

THE TABLEAU

ROW	SLK 3	SLK 4	SLK 5	SLK 6	SLK 7	SLK 8	SLK 9
1	18.007	22.420	24.258	25.024	51.349	2541.6	3600.69
2	-.002	-.001	0.000	0.000	0.000	-.417	-.174
3	-.004	-.002	-.001	0.000	0.000	.000	-.417
4	.000	-.004	-.002	-.001	0.000	.000	.000
5	.000	.000	-.004	-.002	-.001	.000	.000
6	.000	.000	.000	-.004	-.003	.000	.000
7	.000	.000	.000	.000	-.007	.000	.000
8	-.003	-.001	0.000	0.000	0.000	-.625	-.260
9	.004	-.003	-.001	0.000	0.000	.000	-.625
10	.000	.004	-.003	-.001	-.001	.000	.000
11	.000	.000	.004	-.003	-.002	.000	.000
12	.000	.000	.000	.004	-.004	.000	.000
13	.002	.001	0.000	0.000	0.000	.417	.174

THE TABLEAU

ROW	SLK 10	SLK 11	SLK 12	SLK 13	*rhs*
1	0.40E+04	0.42E+04	0.43E+04	.00	-0.14E+07
2	-.072	-.030	-.013	.000	59.087
3	-.174	-.072	-.030	.000	61.810
4	-.417	-.174	-.072	.000	58.343
5	.000	-.417	-.174	.000	70.023
6	.000	.000	-.417	.000	68.056
7	.000	.000	.000	.000	73.333
8	-.109	-.045	-.019	.000	8.631
9	-.260	-.109	-.045	.000	2.714
10	-.625	-.260	-.109	.000	17.514
11	.000	-.625	-.260	.000	5.035
12	.000	.000	-.625	.000	12.083
13	.072	.030	.013	1.000	.913

VARIABLE	*Integer LP Solution*	*Continuous LP Solution*
Y1	9.000000	8.630985
X1	60.000000	59.087322
Y2	3.000000	2.714363
X2	63.000000	61.809574
Y3	18.000000	17.514470
X3	59.000000	58.342980
Y4	6.000000	5.034724
X4	71.000000	70.023150
Y5	12.000000	12.083333
X5	69.000000	68.055560
X6	74.000000	73.333330
OBJECTIVE FUNCTION	1432800.00	1410977.00

A mill of the **Fine-Webb Paper Company** produces "liner board" in jumbo reels with a standard width of 68 inches and a certain fixed length.

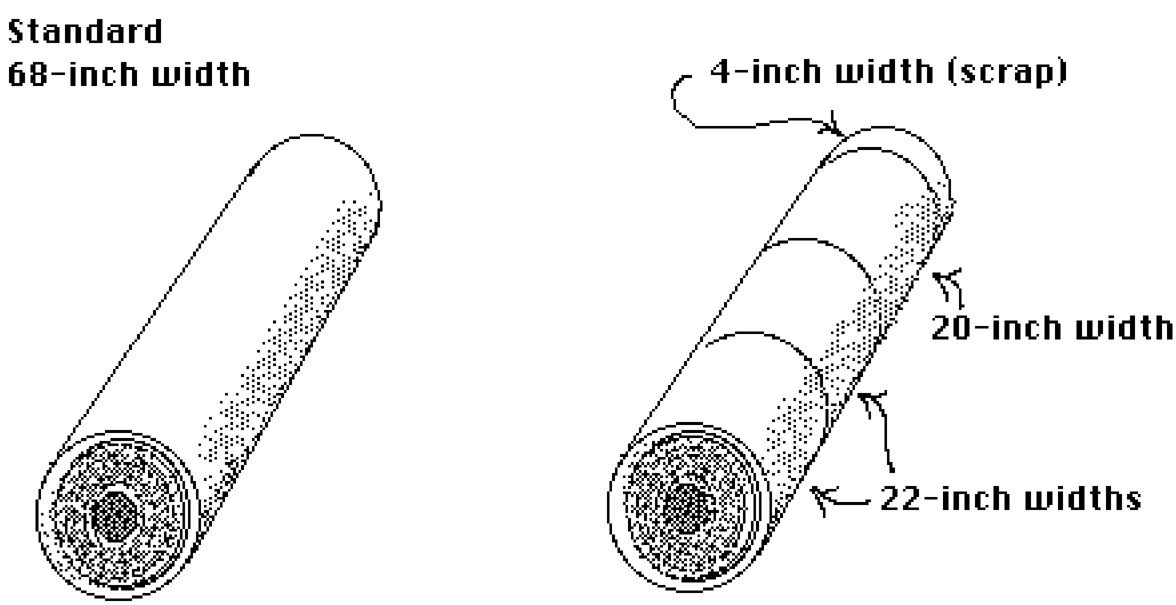
Customers order reels having various smaller widths (but the standard length).

These are to be cut from the standard-width reel, so as to minimize total trim waste (assuming any excess, whatever its width, is to be scrapped).

Today's orders are

110	reels of width	22	inches
120	reels of width	20	inches
80	reels of width	12	inches

Example Cutting Pattern



	PATTERN									
width	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
22"	3	2	2	1	1	1	0	0	0	0
20"	0	1	0	2	1	0	3	2	1	0
12"	0	0	2	0	2	3	0	2	4	5
width of scrap	*2*	*4*	*0*	*6*	*2*	*10*	*8*	*4*	*0*	*8*

VARIABLES

X_i = number of standard rolls cut with pattern i

OBJECTIVE

Minimize amount of scrap, or equivalently,
Minimize number of standard rolls used

(because of assumption that all unneeded rolls are to be scrapped!)

```
MIN      X1 + X2 + X3 + X4 + X5 + X6 + X7 + X8 + X9 + X10
SUBJECT TO
    2)  3 X1 + 2 X2 + 2 X3 + X4 + X5 + X6  >=   110
    3)  X2 + 2 X4 + X5 + 3 X7 + 2 X8 + X9 >=   120
    4)  2 X3 + 2 X5 + 3 X6 + 2 X8 + 4 X9 + 5 X10  >=   80
END

LP OPTIMUM FOUND AT STEP      4

    OBJECTIVE FUNCTION VALUE

    1)    90.0000000
```


Note that we have not imposed any restriction that the variables (# of reels cut with each of the patterns) be integer!

"Picture" of the initial tableau

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10		
1:	1	1	1	1	1	1	1	1	1	1		MIN
2:	3	2	2	1	1	1		'		'	>	C
3:	'	1	'	2	1	'	3	2	1	'	>	C
4:	'		2		2	3		'	4	5	>	B

©D.L.Bricker, U. of Iowa, 1998

VARIABLE	VALUE	REDUCED COST
X1	20.000000	.000000
X2	.000000	.000000
X3	.000000	.000000
X4	50.000000	.000000
X5	.000000	0.000000
X6	.000000	.166667
X7	.000000	0.000000
X8	.000000	0.000000
X9	20.000000	.000000
X10	.000000	.166667

by "chance", the optimal values are integer!

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	-.333333
3)	.000000	-.333333
4)	.000000	-.166667

RANGES IN WHICH THE BASIS IS UNCHANGED

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	1.000000	0.000000	0.000000
X2	1.000000	INFINITY	.000000
X4	1.000000	0.000000	.000000
X5	1.000000	INFINITY	0.000000
X6	1.000000	INFINITY	.166667
X7	1.000000	INFINITY	0.000000
X8	1.000000	INFINITY	0.000000
X10	1.000000	INFINITY	.166667
X3	1.000000	INFINITY	.000000
X9	1.000000	0.000000	.666667

RANGES IN WHICH THE BASIS IS UNCHANGED

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	110.000000	INFINITY	60.000003
3	120.000000	120.000000	100.000000
4	80.000000	400.000000	80.000000

THE TABLEAU

ROW	(BASIS)	X1	X2	X3	X4	X5	X6	X7	X8
1	ART	.0	.000	.000	.0	0.000	.167	0.000	0.000
2	X1	1.0	.500	.750	.0	.250	.458	-.500	-.250
3	X4	.0	.500	-.250	1.0	.250	-.375	1.500	.750
4	X9	.0	.000	.500	.0	.500	.750	.000	.500

ROW	X9	X10	SLK 2	SLK 3	SLK 4	*rhs*
1	.0	.167	.333	.333	.167	-90.000
2	.0	.208	-.333	.167	-.042	20.000
3	.0	-.625	.000	-.500	.125	50.000
4	1.0	1.250	.000	.000	-.250	20.000

There is also a "column-generating" procedure, by Gilmore & Gomory, to automatically generate columns (i.e. cutting patterns).
The procedure requires only a few patterns to be specified at the beginning.
After each simplex pivot, a new pattern is found automatically (based on simplex multipliers) which will have a negative reduced cost, and is pivoted into the basis.

Does not require the user to enumerate all patterns at the beginning!