Farkas' Lemma

Let

\[ A \in \mathbb{R}^{m \times n}, \text{i.e., } A \text{ is } m \times n \text{ matrix,} \]
\[ b \in \mathbb{R}^m, \]
\[ x \in \mathbb{R}^n, y \in \mathbb{R}^m \]

The following statements are equivalent:

1. \[ y^T A \leq 0 \Rightarrow y^T b \leq 0 \]

&

2. \[ \exists x \text{ such that } A x = b, x \geq 0 \]
Proof

Consider the primal/dual LP pair:

\[ \begin{align*}
\text{P} & \quad \text{Minimize } 0x \\
& \quad \text{subject to } A x = b \\
& \quad \text{subject to } x \geq 0 \\
\text{D} & \quad \text{Maximize } y^T b \\
& \quad \text{subject to } A^T y \leq 0, \\
& \quad \text{i.e., } y^T A \leq 0
\end{align*} \]

Problem D is feasible (e.g., let \( y = 0 \), for which the objective \( y^T b \) is zero.)

If statement 1 is true, i.e., \( y^T A \leq 0 \Rightarrow y^T b \leq 0 \)
then \( y = 0 \) must be optimal for problem D.

If \( y = 0 \) is optimal for D, then by LP duality theory, P is feasible (with optimal value 0), proving that \( 1 \Rightarrow 2 \).

Suppose that \( Ax = b \) for some \( x \geq 0 \), and \( y^T A \leq 0 \) for some \( y \).
Then \( y^T A \leq 0 \Rightarrow y^T A x \leq 0 \Rightarrow y^T b \leq 0 \)
proving that \( 2 \Rightarrow 1 \).

QED
Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

The columns of $A$ are points (vectors) in $\mathbb{R}^2$

$A^1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $A^2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

The system $Ax = b$ has a solution if and only if $b$ is a non-negative combination of the columns of $A$, i.e., iff $b$ lies in the cone generated by $A^1$, $A^2$, and $A^3$.

(requirements space)
For example, 

\[
\begin{bmatrix}
1 \\
3
\end{bmatrix}, \quad
\begin{bmatrix}
3 \\
2
\end{bmatrix}, \quad
\begin{bmatrix}
3 \\
1
\end{bmatrix}
\]

\[
b = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 1A^1 + 0A^2 + 1A^3
\]

\[
= \frac{4}{7}A^1 + \frac{4}{7}A^2 + 0A^3
\]

\[
= \frac{11}{14}A^1 + \frac{4}{7}A^2 + \frac{1}{2}A^3
\]

..., etc.

Let \( H_j \) be the hyperplane (a line in \( \mathbb{R}^2 \)) through the origin, orthogonal to \( A^j \), and let \( H_j^c \) be the closed halfspace on the side of \( H_j \) not containing \( A^j \)
\[ y^T A^j = 0 \iff y \perp A^j \]
\[ \iff y \in H_j \]

Also,
\[ y^T A^j \leq 0 \iff y \in H_j \]

Therefore,
\[ y^T A \leq 0 \iff y \in \bigcap_{j} H^{-}_j \]

(the intersection of the half-spaces, shaded above)

Likewise, for a given \( b \), let
\[ H_b = \text{hyperplane through the origin, orthogonal to } b, \text{ and} \]
\[ H^{-}_b = \text{closed half-space on side of } H_b \text{ not containing } b. \]

Then the statement
\[ y^T A \leq 0 \Rightarrow y^T b \leq 0 \]

simply says that
\[ \bigcap_{j} H^{-}_j \subseteq H^{-}_b \]
EXAMPLE 1

\[ b = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \]

Note that \( b \) is in the cone generated by \( A^1, A^2, \) & \( A^3 \) and that \( \bigcap_j H_j \subseteq H_b \)

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EXAMPLE 2

\[ b = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \]

In this case, the vector \( b \) does not lie in the cone generated by \( A \), nor does \( \bigcap_j H_j \) lie entirely in the closed half-space \( H_b \)

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APPLICATION TO NONLINEAR PROGRAMMING

Consider the problem

\[ \begin{align*}
\text{Minimize } & f(x) \\
\text{subject to } & g_i(x) \leq 0, \ i = 1, 2, \ldots, m
\end{align*} \]

Denote

\[ \begin{align*}
b & \equiv -\nabla f(x^*) \\
A^i & \equiv \nabla g_i(x^*) \\
y & \equiv d \quad \text{(direction vector)} \\
x_i & \equiv \lambda_i \quad \text{for } i \in I \equiv \{ i \mid g_i(x^*) = 0 \}
\end{align*} \]

(Lagrange multiplier) \quad \leftrightarrow \quad \text{(index set of tight constraints)}

\[ \Theta \]

Farkas’ Lemma

1. \[ y^T A \leq 0 \Rightarrow y^T b \leq 0 \]

2. \[ \exists x \text{ such that } A x = b, \ x \geq 0 \]

are equivalent statements

That is,

1. \[ d^T \nabla g_i(x^*) \leq 0 \quad \forall i \in I \Rightarrow -d^T \nabla f(x^*) \leq 0 \]

2. \[ \exists \lambda_i \geq 0 \text{ such that } \sum_{i \in I} \lambda_i \nabla g_i(x^*) = -\nabla f(x^*) \]

are equivalent statements
Farkas' Lemma

\[ d^T \nabla g_1(x^*) < 0 \]

\[ \nabla g_3(x^*) \]

\[ x^* \quad g_3(x) = 0 \]

\[ d \text{ is a feasible direction} \]

\[ d^T \nabla g_1(x^*) = 0 \]

\[ \nabla g_3(x^*) \]

\[ x^* \quad g_3(x) = 0 \]

\[ d \text{ is tangent to the constraint boundary} \]

\[ d \text{ is "feasible", but any positive step in this direction may be infeasible} \]

If a constraint is not tight, then any direction is feasible with respect to that constraint!
1. $d^T \nabla g_i(x^*) \leq 0 \quad \forall i \in I \implies -d^T \nabla f(x^*) \leq 0$

directions satisfying $d^T \nabla g_i(x^*) \leq 0 \quad \forall i \in I$
are feasible directions

directions satisfying $d^T \nabla f(x^*) \geq 0$
are directions of ascent

1. Every feasible direction is non-improving

2. $\exists \lambda_i \geq 0$ such that $\sum_{i \in I} \lambda_i \nabla g_i(x^*) = -\nabla f(x^*)$

Steepest descent direction is in the cone generated by gradients of tight constraints

Karush-Kuhn-Tucker Conditions
**K-K-T "Necessary" Condition for Optimality**

If \( x^* \) is an optimal solution to

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0, \; i=1,2,\ldots,m
\end{align*}
\]

then

The directional derivative of \( f(x) \) is nonnegative in every feasible direction at \( x^* \)

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**K-K-T "Necessary" Condition for Optimality**

If \( x^* \) is an optimal solution to

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0, \; i=1,2,\ldots,m
\end{align*}
\]

then

The steepest descent direction at \( x^* \) is in the cone generated by the gradients of the tight constraints at \( x^* \)

©D.L. Enright, U. Equivalent condition, according to Farkas' lemma