

*Extreme
Value
Distributions*



In the design of many engineering systems, we are concerned with the largest or smallest of a set of random variables:

- maximum demand on a production system
 - maximum stress on a system component
 - maximum flood level
 - maximum floor loadings
 - minimum strength of system components
- ... etc.

Let X_1, X_2, \dots, X_n be a set of random variables,
and $Y = \max\{X_1, X_2, \dots, X_n\}$

Then the CDF (cumulative distribution function)
of Y is $F_Y(y) = P\{Y \leq y\} = P\{X_i \leq y \text{ for } i=1,2,\dots,n\}$

If the X_i 's are *independent*,

$$\begin{aligned} F_Y(y) &= P\{X_1 \leq y \ \& \ X_2 \leq y \ \& \ \dots \ \& \ X_n \leq y\} \\ &= P\{X_1 \leq y\} \times P\{X_2 \leq y\} \times \dots \times P\{X_n \leq y\} \\ &= F_{X_1}(y) \times F_{X_2}(y) \times \dots \times F_{X_n}(y) \end{aligned}$$

and if the X_i 's are also *identically distributed*,

$$F_Y(y) = \left(F_X(y) \right)^n$$

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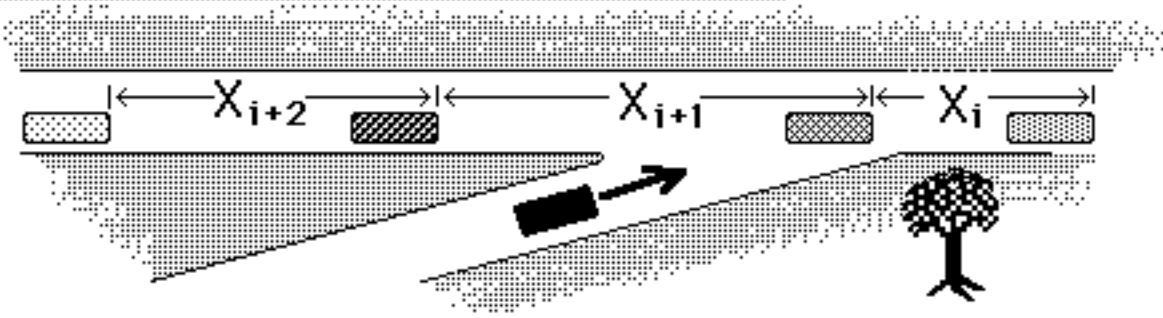
If X_i are continuous, independent, & identically-
distributed random variables with density function
 $f_X(y)$, then

the *density function* for $Y = \max\{X_i\}$ is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy} [F_X(y)]^n \\ &= n [F_X(y)]^{n-1} \frac{d}{dy} F_X(y) \\ &= n [F_X(y)]^{n-1} f_X(y) \end{aligned}$$

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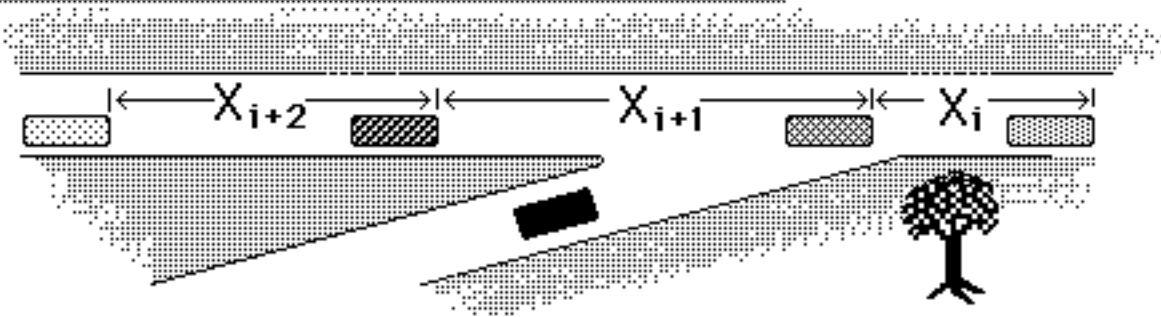
Example: Merging Traffic



A certain class of drivers will merge into freely flowing traffic *only if* the time X_i between oncoming cars is *at least* γ seconds.

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Example: Merging Traffic



Assume that the X_i 's are i.i.d. random variables having an exponential distribution with parameter λ (the arrival rate).

What is the probability that a driver will NOT have merged after n cars have passed?



Solution

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Example: Merging Traffic

Let $Y = \max\{X_1, X_2, \dots, X_n\}$

Then $P\{\text{no merge after } n \text{ cars}\} = P\{Y \leq y\}$

$$\begin{aligned} &= F_Y(y) = [F_X(y)]^n \\ &= (1 - e^{-\lambda y})^n \end{aligned}$$

and the density function of Y is

$$f_Y(y) = n\lambda (1 - e^{-\lambda y})^{n-1} e^{-\lambda y} \quad (y \geq 0)$$



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Suppose that the rate of traffic is $\lambda = \frac{1 \text{ car}}{5 \text{ seconds}}$

and $y=10$ seconds between oncoming cars is required for merging.

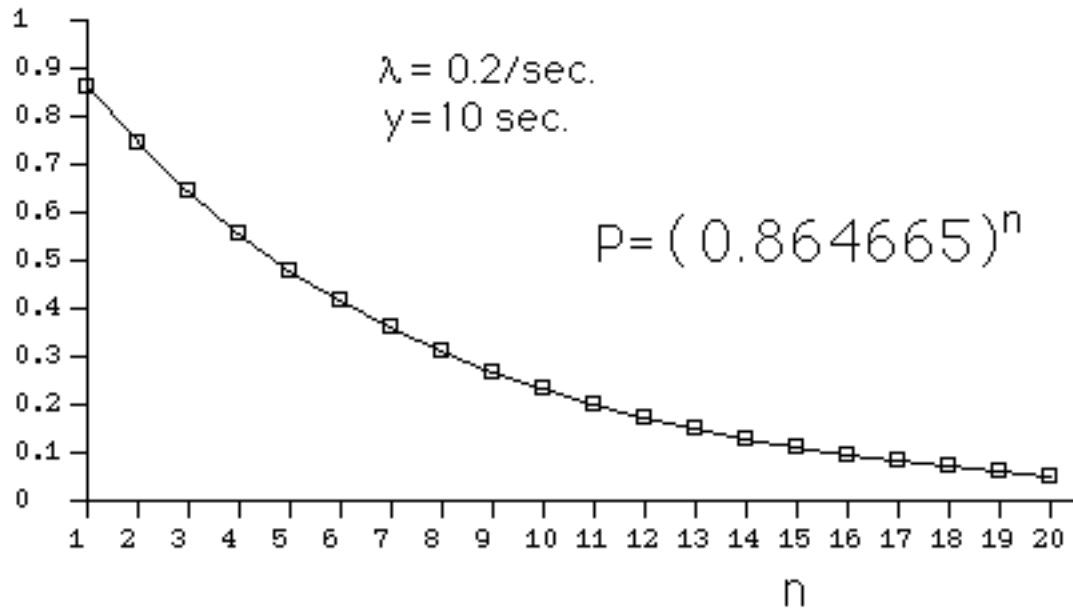
Then

$P\{\text{no merge after } n \text{ cars have passed}\}$

$$\begin{aligned} &= (1 - e^{-\lambda y})^n = (1 - e^{-2})^n \\ &= (0.864665)^n \end{aligned}$$

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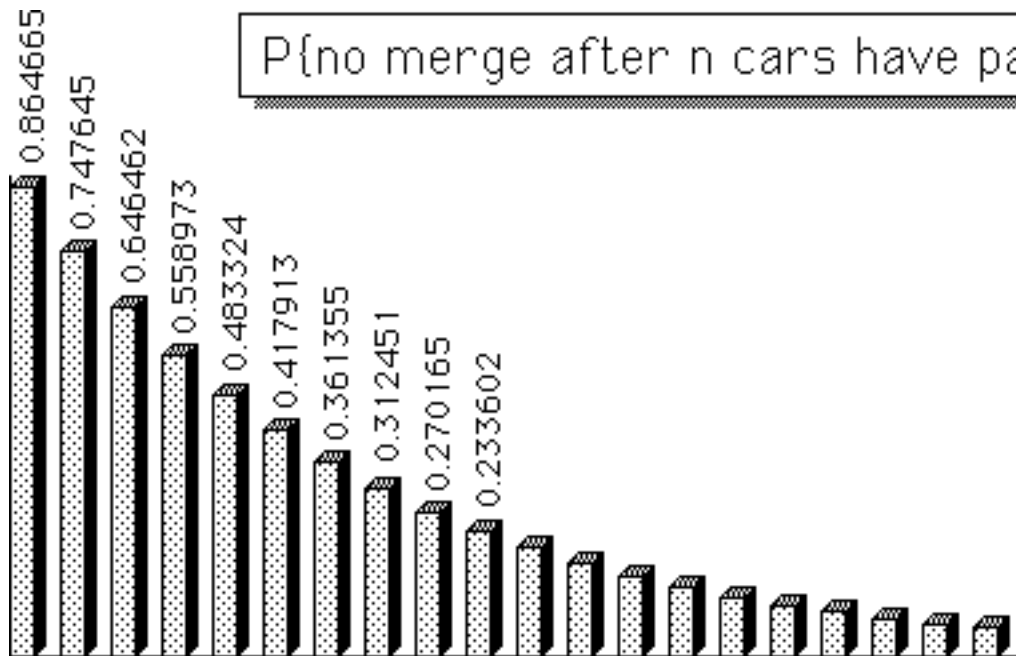
P{no merge after n cars have passed}



n	P
1	0.864665
2	0.747645
3	0.646462
4	0.558973
5	0.483324
6	0.417913
7	0.361355
8	0.312451
9	0.270165
10	0.233602

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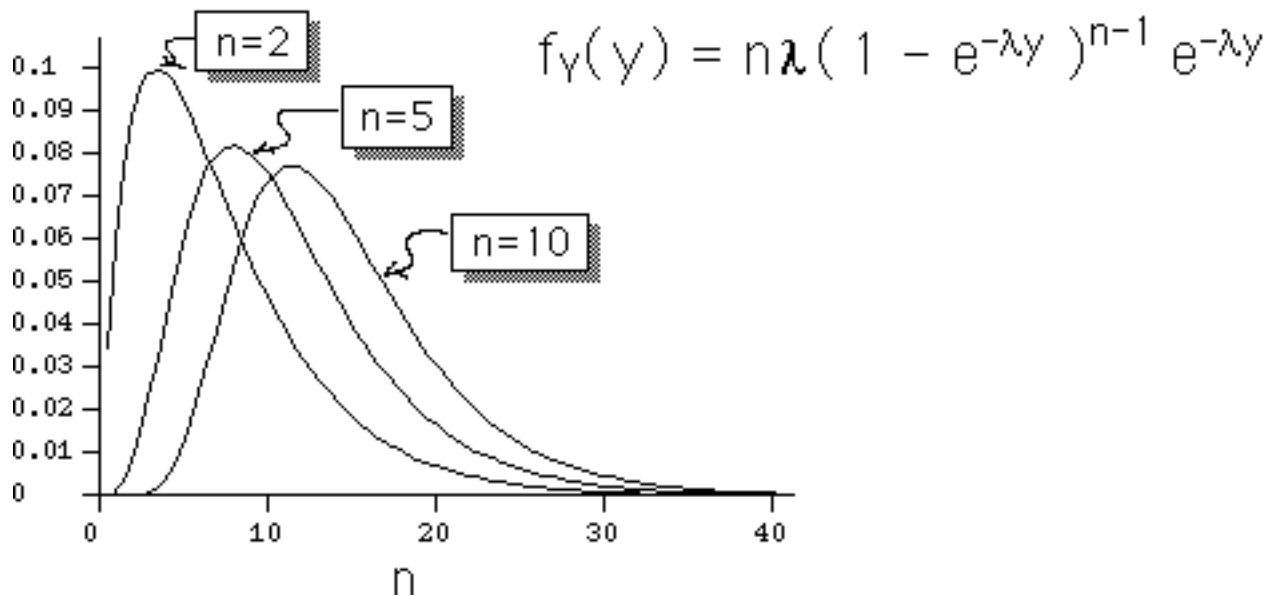
P{no merge after n cars have passed}



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Extreme Value density functions

$\lambda = 0.2/\text{sec.}$
 $y = 10 \text{ sec.}$



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Asymptotic Distributions

From the *Central Limit Theorem* we know that, for "large" n ,

$Y = \sum_{i=1}^n X_i$ has approximately a *Normal* distribution

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\}$$

$Y = \prod_{i=1}^n X_i$ has approximately a *Lognormal* dist'n

$$f_Y(y) = \frac{1}{y\sigma_Y\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(Y/\mu)}{\sigma}\right)^2\right\}$$

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Asymptotic Distributions

Are there such limiting distributions for

$$\max\{X_1, X_2, \dots, X_n\}$$

and

$$\min\{X_1, X_2, \dots, X_n\}$$

as $n \rightarrow \infty$?

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The limiting distribution for Y depends upon the shape of the tail of the density functions f_X .

- ☞ Type 1: f_X falls off in "exponential" manner in the tail of interest
- ☞ Type 3: f_X falls to zero in the tail of interest

"tail of interest" is right for $Y = \max\{X_i\}$,
and left for $Y = \min\{X_i\}$.

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Type 1

Maximum of X_i 's, which have a common CDF of the form

$$F_x(x) = 1 - e^{-g(x)}$$

where $g(x)$ is an increasing function of x .

(For example, if exponential distribution, $g(x) = \lambda x$. The normal and gamma distributions are also of this type.)



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The limiting distribution of $\max\{X_1, X_2, \dots, X_n\}$ is the *Gumbel* distribution:

CDF $F_Y(y) = \exp[-e^{-\alpha(y-u)}]$

density $f_Y(y) = \alpha \exp[-\alpha(y-u) - e^{-\alpha(y-u)}]$

with

mean $\mu_Y \approx u + \frac{0.577}{\alpha}$

std. deviation $\sigma_Y \approx \frac{1.282}{\alpha}$

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Example - Gumbel Distn.

An engineer has good estimates for the mean and standard deviation of the peak annual flow in a small stream:

(Peak annual flow = largest of 365 daily flows)

mean: $\mu_Y = 100$ cfs (ft³/sec.)

st'd dev'n: $\sigma_Y = 50$ cfs.

What is the probability that, next year, the peak flow will exceed 200 cfs?



Solution

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Since the peak annual flow is the maximum of a large number (365) daily flows, we will assume a *Gumbel* distribution.

First we use the mean and standard deviation to compute the parameters α and u :

$$\left. \begin{aligned} \sigma_Y &\approx \frac{1.282}{\alpha} \Rightarrow \alpha \approx \frac{1.282}{\sigma_Y} = 0.0256 \\ \mu_Y &\approx u + \frac{0.577}{\alpha} \Rightarrow u \approx \mu_Y - \frac{0.577}{\alpha} \end{aligned} \right\} u = 77.5 \text{ cfs}$$



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$$F_Y(y) = \exp[-e^{-\alpha(y-u)}], \alpha = 0.256, u = 77.5$$

We can now compute the probability that the peak flow in a particular year will exceed 200 cfs:

$$\begin{aligned} P\{Y \geq 200\} &= 1 - F_Y(200) \\ &= 1 - \exp\{-e^{-0.256(200-77.5)}\} \\ &= 0.043 \end{aligned}$$

That is, a flow exceeding 200 cfs will occur once every $\frac{1}{0.043} \approx 23$ years.

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Example

The annual maximum rate of flow of a particular river has a mean of 10K cfs with standard deviation of 3K cfs. Assume that this maximum rate of flow has a Gumbel distribution.

- *What are the parameters of this distribution?*
- *Compute $P\{\text{annual max flowrate} \geq 15\text{K cfs}\}$*



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- Find an expression for the CDF of the river's maximum flow rate over the 20 year lifetime of an anticipated flood-control project. (Assume that the individual annual max flow rates are i.i.d. with Gumbel distribution, as before.)
- Compute $P\{20\text{-year maximum flow rate} \geq 15\text{K cfs}\}$



Solution

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Parameters of the Distribution

$$\sigma_Y \approx \frac{1.282}{\alpha} \Rightarrow \alpha \approx \frac{1.282}{\sigma_Y} = \frac{1.282}{3} = 0.4273333$$

$$\mu_Y \approx u + \frac{0.577}{\alpha} \Rightarrow u \approx \mu_Y - \frac{0.577}{\alpha} = 10 - \frac{0.577}{0.42733}$$

$$u = 8.649766$$



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$$F_Y(y) = \exp[-e^{-\alpha(y-u)}], \alpha = 0.4273, u = 8.64977$$

P{annual max flowrate \geq 15K cfs} = ?

t	F(t)	1-F(t)
10	0.57030	0.42969
11	0.69330	0.30669
12	0.78748	0.21251
13	0.85570	0.14429
14	0.90335	0.09664
15	0.93585	0.06414
16	0.95768	0.04231
17	0.97219	0.02780
18	0.98177	0.01822
19	0.98807	0.01192
20	0.99220	0.00779

The annual peak flow will exceed 15K cfs with probability approximately 6.4%

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Let $Y = \text{Max}\{X_1, X_2, \dots, X_{20}\}$,

where $X_i =$ peak flowrate in year i .

Each random variable X_i is assumed to have a Gumbel distribution:

$$F_X(t) = \exp[-e^{-0.4273(t-8.64977)}]$$

The 20-year maximum flowrate will therefore have the CDF:

$$\begin{aligned} F_Y(t) &= [F_X(t)]^{20} \\ &= \left\{ \exp[-e^{-0.4273(t-8.64977)}] \right\}^{20} \end{aligned}$$

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$$\begin{aligned} & P\{20\text{-year maximum flow rate} \geq 15\text{K cfs}\} \\ &= 1 - F_Y(15) \\ &= 1 - [F_X(15)]^{20} \\ &= 1 - [0.93585]^{20} \\ &= 1 - 0.26557 \\ &= 0.73443 \end{aligned}$$

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Example

It has been verified experimentally that the velocity of an arbitrary wind gust has an exponential distribution, and hence a rapid convergence to a Gumbel distribution should be expected for the maximum gust velocity occurring during a thunderstorm.

This maximum gust velocity has an estimated mean of 15.6 ft/sec. with a standard deviation of 6.2 ft/sec.



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What is the probability that the maximum gust velocity during a thunderstorm exceeds 30 mph?

What is the probability that the maximum gust velocity will be less than 10 mph?

Solution

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Gumbel Distribution

Parameters: U= 12.809516, alpha= 0.20677419

$P\{Y \leq 10\} = 16.7\%$

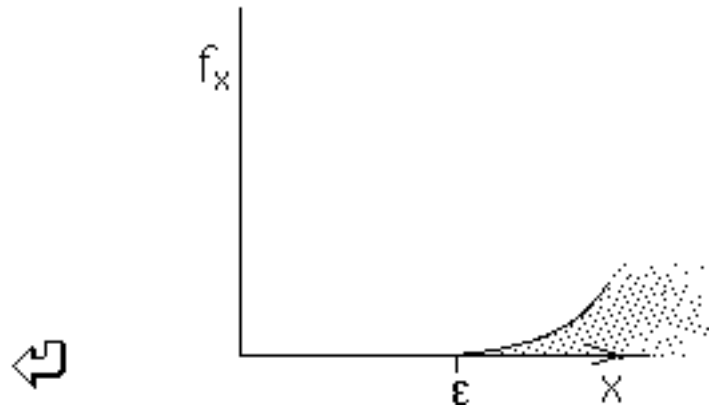
t	F(t)	1-F(t)
5	0.006559	0.993440
10	0.167342	0.832657
15	0.529533	0.470466
20	0.797643	0.202356
25	0.922742	0.077257
30	0.971810	0.028189
35	0.989882	0.010117
40	0.996390	0.003609
45	0.998714	0.001285
50	0.999542	0.000457

$P\{Y \geq 30\} = 2.8\%$



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Type 3 density falls to zero in tail of interest, e.g., if $\min\{X_i\}$, $f_x(x) = 0$ for $x \leq \epsilon$ where ϵ is the *lower bound* on X_i and $F_x(x) \approx c(x-\epsilon)^k$ for x near ϵ and $x \geq \epsilon$
 Assume that X 's are i.i.d.



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The limiting distribution for $Y = \min\{X_i\}$ as $n \rightarrow \infty$ is

CDF $F_Y(y) = 1 - \exp \left[- \left(\frac{y-\epsilon}{u-\epsilon} \right)^k \right]$

density function $f_Y(y) = \frac{k}{u-\epsilon} \left(\frac{y-\epsilon}{u-\epsilon} \right)^{k-1} \exp \left[- \left(\frac{y-\epsilon}{u-\epsilon} \right)^k \right]$

mean $\mu_Y = \epsilon + (u-\epsilon) \Gamma \left(1 + \frac{1}{k} \right)$

variance $\sigma_Y^2 = (u-\epsilon)^2 \left[\Gamma \left(1 + \frac{2}{k} \right) - \Gamma^2 \left(1 + \frac{1}{k} \right) \right]$

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If $\epsilon = 0$, this simplifies to

CDF

$$F_Y(y) = 1 - e^{-(y/u)^k}$$

mean

$$\mu_Y = u \Gamma\left(1 + \frac{1}{k}\right)$$

variance

$$\sigma_Y^2 = u^2 \left\{ \Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right\}$$

which is the *Weibull* distribution.

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The "Gamma" function is defined for all nonnegative x and satisfies

$$\Gamma(1+x) = x!$$

for integer values of x .

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Values of $\Gamma\left(1+\frac{1}{k}\right)$ for k=0.1 through 9.9

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	∞	362880	120.0000	9.2605	3.3234	2.0000	1.5046	1.2658	1.1330	1.0522
1	1.0000	0.9649	0.9407	0.9236	0.9114	0.9027	0.8966	0.8922	0.8893	0.8874
2	0.8862	0.8857	0.8856	0.8859	0.8865	0.8873	0.8882	0.8893	0.8905	0.8917
3	0.8930	0.8943	0.8957	0.8970	0.8984	0.8997	0.9011	0.9025	0.9038	0.9051
4	0.9064	0.9077	0.9089	0.9102	0.9114	0.9126	0.9137	0.9149	0.9160	0.9171
5	0.9182	0.9192	0.9202	0.9213	0.9222	0.9232	0.9241	0.9251	0.9260	0.9269
6	0.9277	0.9286	0.9294	0.9302	0.9310	0.9318	0.9325	0.9333	0.9340	0.9347
7	0.9354	0.9361	0.9368	0.9375	0.9381	0.9387	0.9394	0.9400	0.9406	0.9412
8	0.9417	0.9423	0.9429	0.9434	0.9439	0.9445	0.9450	0.9455	0.9460	0.9465
9	0.9470	0.9474	0.9479	0.9484	0.9488	0.9492	0.9497	0.9501	0.9505	0.9509

E.g., $\Gamma\left(1+\frac{1}{2.5}\right) = \Gamma(1.4) = 0.8873$

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Values of $\Gamma\left(1+\frac{2}{k}\right)$ for k=0.1 through 9.9

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	∞	2.433E18	3628800	2593.6	120.0000	24.0000	9.2605	5.0291	3.3234	2.4786
1	2.0000	1.7024	1.5046	1.3663	1.2658	1.1906	1.1330	1.0880	1.0522	1.0234
2	1.0000	0.9808	0.9649	0.9517	0.9407	0.9314	0.9236	0.9170	0.9114	0.9067
3	0.9027	0.8994	0.8966	0.8942	0.8922	0.8906	0.8893	0.8882	0.8874	0.8867
4	0.8862	0.8859	0.8857	0.8856	0.8856	0.8857	0.8859	0.8862	0.8865	0.8868
5	0.8873	0.8877	0.8882	0.8887	0.8893	0.8899	0.8905	0.8911	0.8917	0.8923
6	0.8930	0.8936	0.8943	0.8950	0.8957	0.8963	0.8970	0.8977	0.8984	0.8991
7	0.8997	0.9004	0.9011	0.9018	0.9025	0.9031	0.9038	0.9044	0.9051	0.9058
8	0.9064	0.9070	0.9077	0.9083	0.9089	0.9096	0.9102	0.9108	0.9114	0.9120
9	0.9126	0.9132	0.9137	0.9143	0.9149	0.9154	0.9160	0.9166	0.9171	0.9176

E.g., $\Gamma\left(1+\frac{2}{2.5}\right) = \Gamma(1.8) = 0.9314$

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Computing Weibull Parameters

Given μ_Y & σ_Y^2 , we wish to solve for u & k :

$$\begin{cases} \mu_Y = u \Gamma\left(1 + \frac{1}{k}\right) \\ \sigma_Y^2 = u^2 \left\{ \Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right\} \end{cases}$$

This is a system of nonlinear equations with 2 unknowns, which might be solved by the Newton-Raphson method.

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Solve for u in terms of the mean:

$$\mu_Y = u \Gamma\left(1 + \frac{1}{k}\right) \Rightarrow u = \frac{\mu_Y}{\Gamma\left(1 + \frac{1}{k}\right)}$$

Eliminate the variable u from the second equation:

$$\begin{aligned} \sigma_Y^2 &= u^2 \left\{ \Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right\} \\ \Rightarrow \sigma_Y^2 &= \frac{\mu_Y^2}{\Gamma^2\left(1 + \frac{1}{k}\right)} \left\{ \Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right\} \end{aligned}$$

This leaves us with a single nonlinear equation in k .

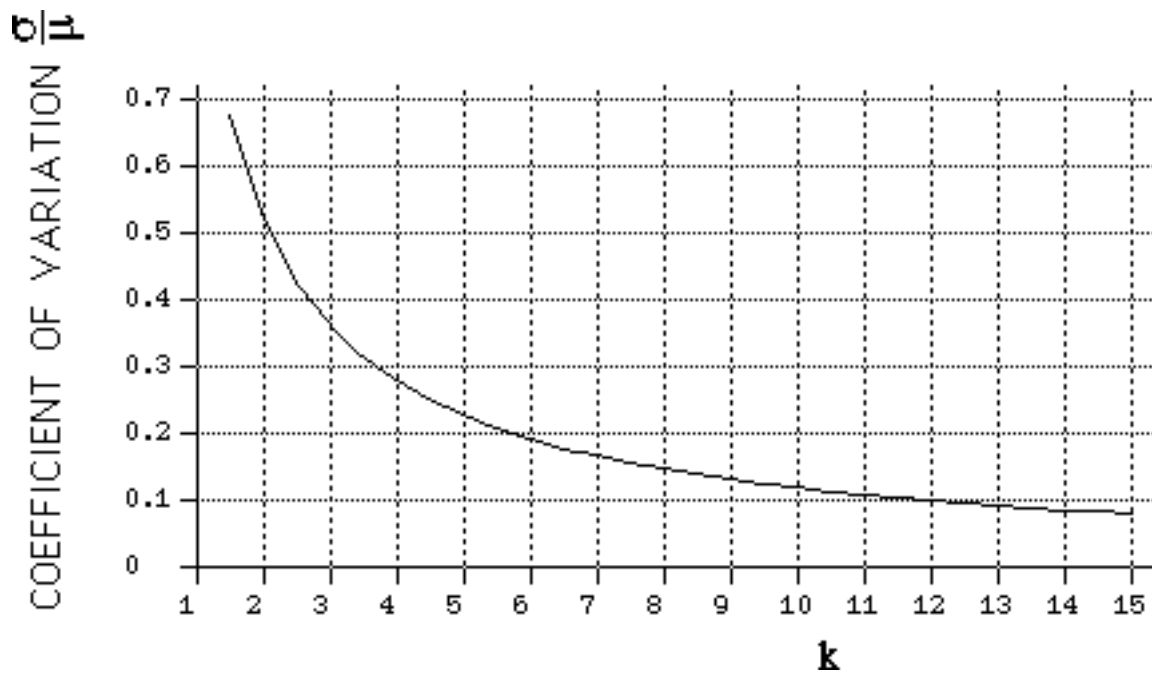
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$$\sigma_Y^2 = \frac{\mu_Y^2}{\Gamma^2\left(1+\frac{1}{k}\right)} \left\{ \Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right) \right\} \Rightarrow \frac{\sigma_Y^2}{\mu_Y^2} = \frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)} - 1$$

$$\Rightarrow \frac{\sigma_Y}{\mu_Y} = \sqrt{\frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)} - 1}$$

Coefficient
of Variation

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Coefficient of Variation $\frac{\sigma}{\mu}$

k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	—	—	15.8	5.41	3.14	2.24	1.76	1.46	1.26	1.11
1	1	0.91	0.837	0.776	0.724	0.679	0.64	0.605	0.575	0.547
2	0.523	0.5	0.48	0.461	0.444	0.428	0.413	0.399	0.387	0.375
3	0.363	0.353	0.343	0.334	0.325	0.316	0.309	0.301	0.294	0.287
4	0.281	0.274	0.268	0.263	0.257	0.252	0.247	0.242	0.238	0.233
5	0.229	0.225	0.221	0.217	0.213	0.21	0.206	0.203	0.2	0.197
6	0.194	0.191	0.188	0.185	0.183	0.18	0.177	0.175	0.173	0.17
7	0.168	0.166	0.164	0.162	0.16	0.158	0.156	0.154	0.152	0.15
8	0.148	0.147	0.145	0.143	0.142	0.14	0.139	0.137	0.136	0.134
9	0.133	0.131	0.13	0.129	0.128	0.126	0.125	0.124	0.123	0.121

*for example, if $k=2.5$,
coefficient of variation
is 0.428*

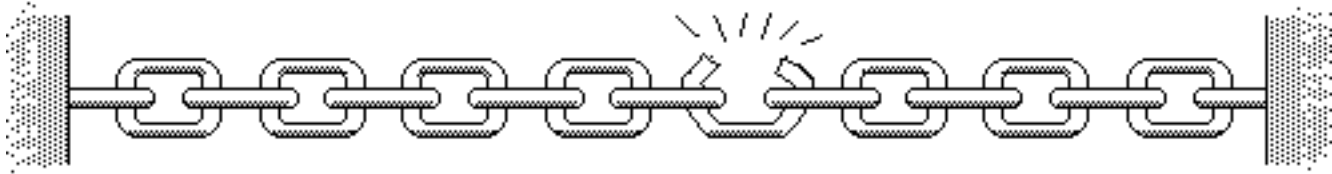
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Given the coefficient of variation ($\frac{\sigma}{\mu}$), we can either approximate k through use of the table or graph, or we can use a numerical method (e.g., the secant method) to solve the nonlinear equation

$$\frac{\sigma_Y}{\mu_Y} = \sqrt{\frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)} - 1}$$

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Example - Weibull Dist'n



Let X_i = lifetime of link i of chain ($X_i \geq 0$)
 $Y = \min\{X_1, X_2, \dots, X_n\}$ = lifetime of chain

For large n (long chains), the distribution of Y should be approximately Weibull.

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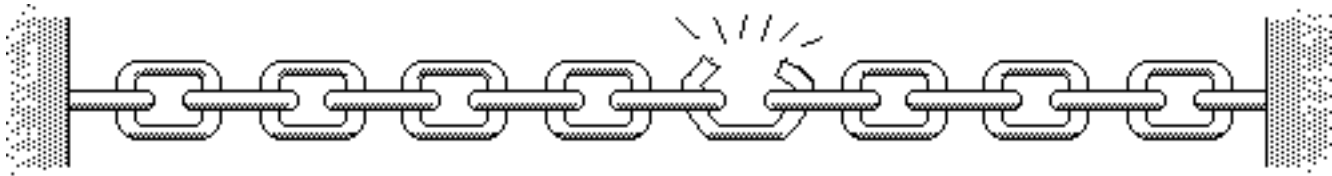
Suppose that we have estimates for the mean lifetime and standard deviation:

$$\mu_Y = 150 \text{ hours} \quad \sigma_Y = 50 \text{ hours}$$

What is the probability that the chain...

- fails before 100 hours of use?
- has not yet failed after 200 hours of use?

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Since the lifetime of the chain is the minimum of the lifetimes (times until failure) of the individual links of the chain, and these lifetimes are each bounded below by zero, we will assume the *Weibull* distribution.

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Computation of Parameter k

The coefficient of variation is $\frac{\sigma}{\mu} = \frac{1}{3}$

We will use the *Secant Method* to solve the nonlinear equation:

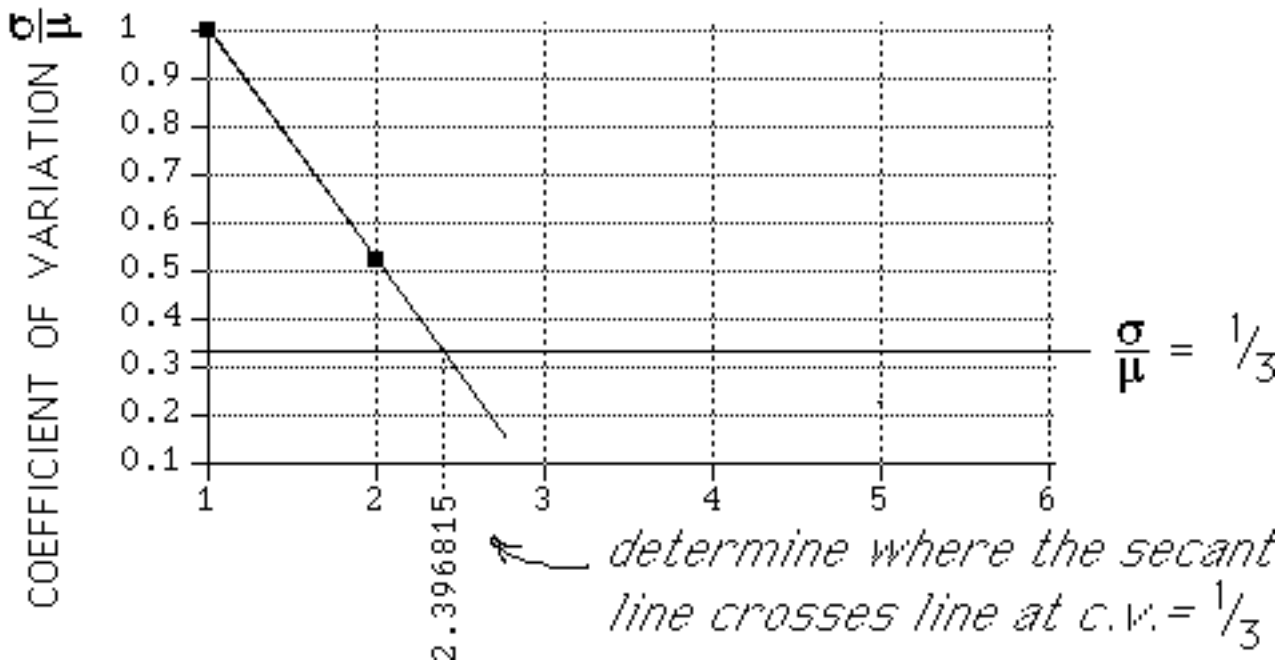
$$\frac{\sigma_Y}{\mu_Y} = \sqrt{\frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)} - 1}$$

for the parameter k

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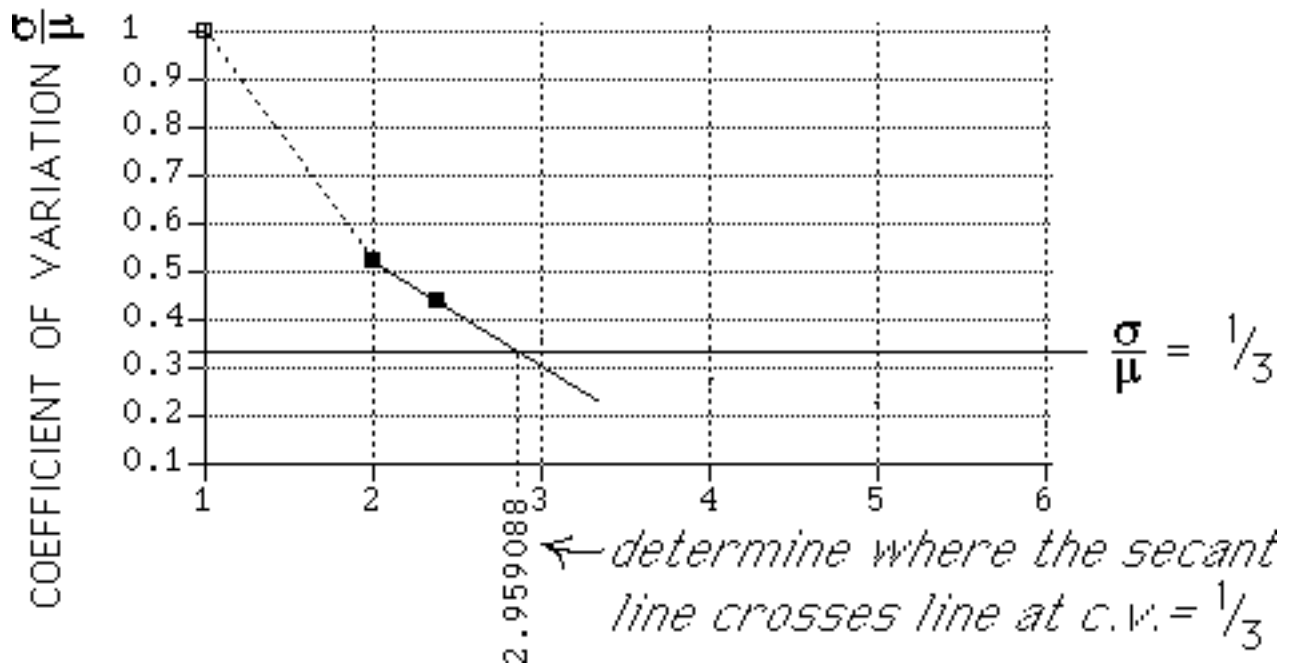
Secant method

Start with "guesses" $k=1$ & 2



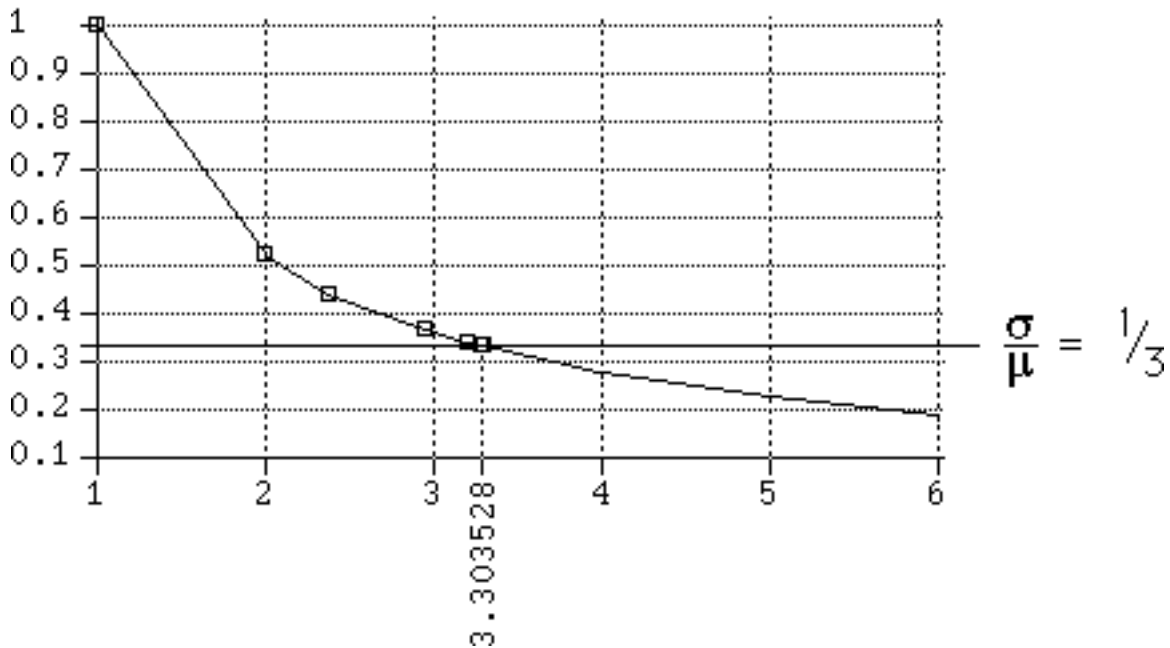
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Using last 2 values of k , draw a new secant line



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Continue, until the procedure converges to $k = 3.303528$



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Using the secant method, we get the following approximations to the value of k :

k	error
1.000000	0.666667
2.000000	0.189390
2.396815	0.111032
2.959088	0.034614
3.213779	0.008349
3.294735	0.000799
3.303305	0.000020
3.303528	0.000000

(6 iterations were performed!)

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Given $k = 3.303528$ and mean $\mu_Y = 150$, we can now solve for the parameter u :

$$u = \frac{\mu_Y}{\Gamma\left(1 + \frac{1}{k}\right)} = \frac{150}{\Gamma(1.3027)} = \frac{150}{0.897065} = 167.212$$

What is the probability that the chain...

- fails before 100 hours of use?
- has not yet failed after 200 hours of use?

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t	F(t)	1-F(t)	
100	0.16722352	0.83277648	<div style="border: 1px solid black; padding: 5px; width: fit-content; margin-bottom: 10px;"> $k = 3.303528$ $u = 167.212$ </div> <p><i>Probability that chain fails before 100 hours is approx. 16.7%</i></p>
110	0.2217509	0.7782491	
120	0.28408854	0.71591146	
130	0.35296437	0.64703563	
140	0.42656811	0.57343189	
150	0.502656	0.497344	
160	0.57871986	0.42128014	
170	0.65220465	0.34779535	
180	0.72074716	0.27925284	
190	0.78240031	0.21759969	
200	0.83580659	0.16419341	<p><i>Probability that chain has not failed after 200 hrs. is 16.4%</i></p>

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Cumulative Distribution Function

