Classification of States of a Markov chain

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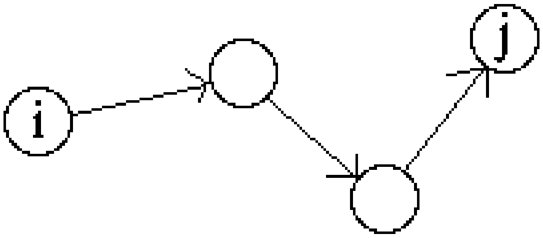
A state i is *recurrent* if, given that the Markov chain starts in state i, the probability that it eventually returns to state i is one.

i.e., $$\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$$

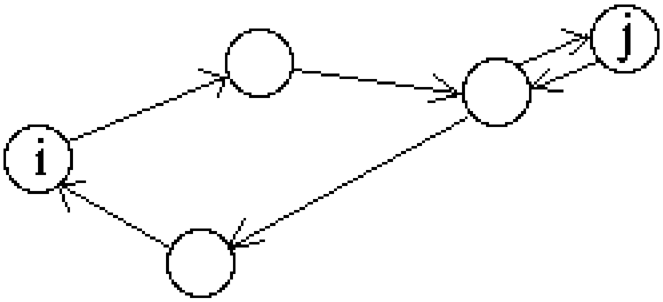
$f_{ij}^{(n)}$ = Probability that the first visit to state **j** occurs at stage **n**, given that the initial state is **i**.

A state which is not recurrent is said to be *transient*.



State j is ***reachable*** from state i

$i \longrightarrow j$



States i & j ***communicate***

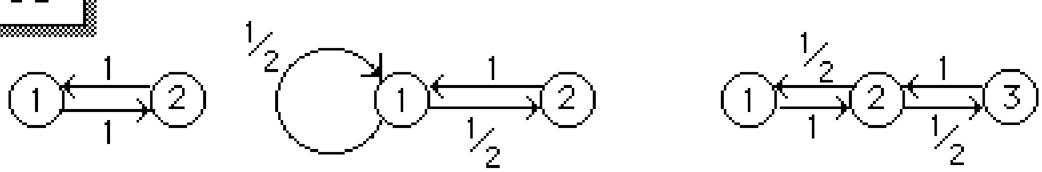
$i \longleftrightarrow j$

If state i is recurrent, and states i & j communicate, then state j is recurrent.

The *period* $d(i)$ of state i is the greatest common divisor of all the integers $n \geq 1$ for which

$$p_{ii}^{(n)} > 0$$

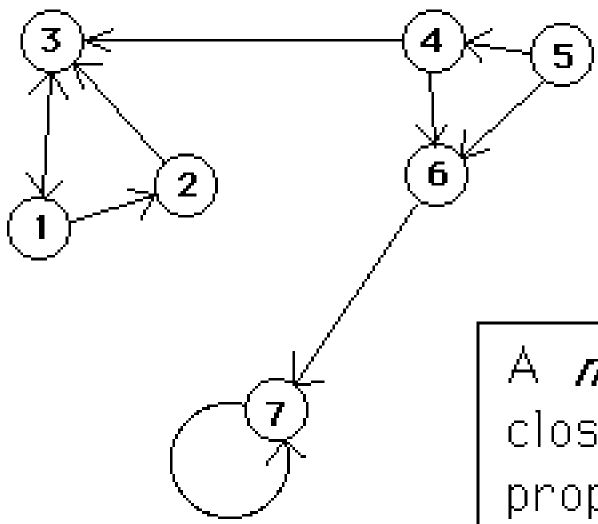
Examples



d(1)=d(2)=2 d(1)=d(2)=1 d(1)=d(2)=d(3)=2

If $i \longleftrightarrow j$, then $d(i)=d(j)$.

A Markov chain with $d(i)=1$ for all i is called *aperiodic*



A set of states is ***closed*** if no state not in the set is reachable from a state in the set

A ***minimal closed set*** is a closed set which has no closed proper subsets.

The closed sets are

{1,2,3,4,5,6,7}
{1,2,3}
{1,2,3,4,6,7}
{7}

both these closed sets are minimal!

A minimal closed set is said to be ***irreducible.***

A Markov chain is called ***irreducible*** if the set of its states is a minimal closed set.

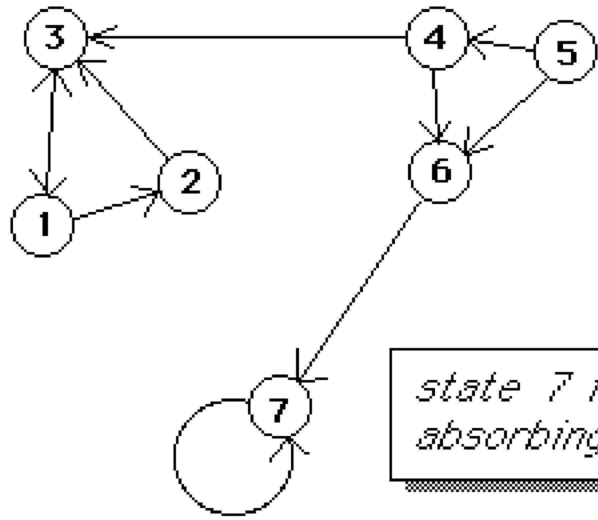
(A Markov chain is ***irreducible*** if and only if every pair of its states communicate.)

A state which forms a closed set, i.e., which cannot reach another state, is said to be ***absorbing.***

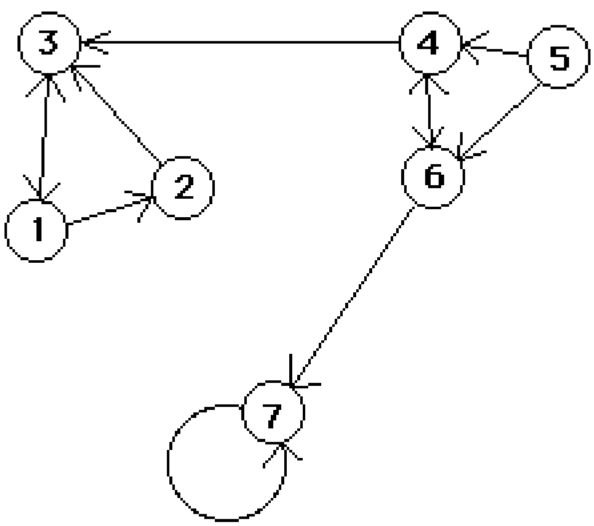
If state j is absorbing, then

$$p_{jj} = p_{jj}^{(n)} = 1$$

for all n=1, 2, ...



state 7 is absorbing.



In a Markov chain with finitely many states, a member of a minimal closed set is ***recurrent*** and other states are ***transient***

States 1,2,3, & 7 are recurrent.

If state j is recurrent, but

$$\lim_{n \to \infty} \mathbf{p}_{ij}^{(n)} = \mathbf{0} \quad \text{for any state i,}$$

then state j is said to be ***null***.

> An irreducible Markov chain with *finitely* many states has
> - **no** recurrent null states
> - **no** transient states

Absorption Analysis

Consider a Markov chain with N states:

- r absorbing states
- s = N−r transient states

Partition the transition probability matrix P:

$$\mathbf{P} = \begin{array}{c} \left[\begin{array}{c:c} \mathrm{I} & 0 \\ \hdashline \mathrm{R} & \mathrm{Q} \end{array}\right] \\ \underbrace{\hphantom{R}}_{\substack{r \\ \textit{columns}}} \; \underbrace{\hphantom{Q}}_{\substack{s \\ \textit{columns}}} \end{array} \begin{array}{l} \} \; \textit{r rows} \\ \} \; \textit{s rows} \\ \end{array}$$

The Powers of P

$$P = \left[\begin{array}{c:c} I & 0 \\ \hdashline R & Q \end{array}\right]$$

$$P^2 = \left[\begin{array}{c:c} I & 0 \\ \hdashline R+QR & Q^2 \end{array}\right], \qquad P^3 = \left[\begin{array}{c:c} I & 0 \\ \hdashline R+QR+Q^2R & Q^3 \end{array}\right]$$

$\vdots$

$$P^n = \underbrace{\left[\begin{array}{c:c} I & 0 \\ \hdashline \left(I+Q+Q^2+\ldots+Q^{n-1}\right)R & Q^n \end{array}\right]}_{\textit{absorbing} \quad\quad \textit{transient}} \begin{array}{l} \} \ \textit{absorbing} \\ \} \ \textit{transient} \end{array}$$

Let states i & j both be transient, and define

e_{ij} = expected # of visits to state j, given that the system begins in state i (counting initial visit if i=j)

$$e_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

and the $r \times r$ matrix:

$$E = \sum_{n=0}^{\infty} Q^n = (I - Q)^{-1}$$

since $(I - Q)\left(I + Q + Q^2 + \ldots\right) = I + Q - Q + Q^2 - Q^2 + \ldots = I$

Absorption probability

Let state i be transient and state j absorbing, and define:

$\mathbf{a_{ij}}$ = probability that the system enters the absorbing state j at some future time, given that it is initially in transient state i

$$\boxed{a_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}}$$

absorption probability (an infinite sum)

An alternate method for computing these probabilities:

Condition on the state entered at stage #1:

$$a_{ij} = \sum_{k=1}^{N} P\{\text{system enters state } j \mid X_1 = k\}\ P\{X_1 = k\}$$

$$= P\{\text{system enters state } j \mid X_1 = j\}\ P\{X_1 = j\}$$

$$+ \sum_{k \text{ absorbing}, \neq j} P\{\text{system enters state } j \mid X_1 = k\}\ P\{X_1 = k\}$$

$$+ \sum_{k \text{ transient}} P\{\text{system enters state } j \mid X_1 = k\}\ P\{X_1 = k\}$$

$$= 1 p_{ij} + 0 + \sum_{k=1}^{s} a_{kj}\, p_{ik}$$

$$\boxed{a_{ij} = p_{ij} + \sum_{k=1}^{s} a_{kj}\, p_{ik}}$$

$$\boxed{a_{ij} = p_{ij} + \sum_{k=1}^{s} a_{kj}\, p_{ik}}$$, i transient, j absorbing

In matrix form:

$$A = R + QA \quad \textit{where} \quad \mathbf{P} = \begin{array}{cc} \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{R} & \mathbf{Q} \end{array}\right] & \begin{array}{l} \}\ \textit{absorbing} \\ \}\ \textit{transient} \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{\textit{absorbing}} & \underbrace{\quad}_{\textit{transient}} \end{array} & \end{array}$$

$$A - QA = R$$

$$(\mathbf{I} - Q)A = R$$

$$A = (\mathbf{I} - Q)^{-1}R$$

$$\boxed{A = ER}$$