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Given: M candidate locations, N customers
\[ F_i = \text{fixed cost of establishing a plant at site } i, \quad i=1,2,...M \]
\[ C_{ij} = \text{cost of supplying all demand of customer } j \text{ from plant } i, \quad j=1,2,...N \]

The Problem: Select a set of plant locations and allocation of customers to plants so as to minimize the total cost.

Note: there are no capacity constraints for a plant which has been selected, and the number of plants is not specified (unlike p-median problem)
ILP models of the SPL problem

Define variables:

\[ Y_i = \begin{cases} 
1 & \text{if plant site } i \text{ is selected} \\
0 & \text{otherwise} 
\end{cases} \]

\[ X_{ij} = \begin{cases} 
1 & \text{if plant } i \text{ serves all demand of customer } j \\
0 & \text{otherwise} 
\end{cases} \]
**Model #1**

Minimize \[ \sum_{i=1}^{M} \sum_{j=1}^{N} C_{ij} X_{ij} + \sum_{i=1}^{M} F_i Y_i \]

s.t. \[ \sum_{i=1}^{M} X_{ij} = 1 \quad \forall \ j=1, \ldots, N \]

\[ X_{ij} \leq Y_i \quad \forall \ i \text{&} j \]

\[ Y_i \in \{0,1\}, \ X_{ij} \geq 0 \quad \forall \ i \text{&} j \]

**Model #2**

Replace constraints \[ X_{ij} \leq Y_i \quad \forall \ i \text{&} j \]

with aggregated constraints

\[ \sum_{j=1}^{N} X_{ij} \leq N Y_i \quad \forall \ i \]
Models #1 & #2 are equivalent, in that the feasible solution sets are identical.... But-- their LP relaxations (i.e., replacing $Y_i \in \{0,1\}$ with $0 \leq Y_i \leq 1$) are not!
**Example**

Minimize $-2X_{i1} - X_{i2}$

Cost = -3

Cost = -4

Feasible set for $X_{i1} + X_{i2} \leq 2$

Feasible set for $X_{i1} \leq 1$

$X_{i2} \leq 1$

Model #1 provides a higher, "better" lower bound on the optimum!

Model #2 is more "compact", and the LP relaxation is easier to solve.

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LP Relaxation of Model #2

At the LP optimum,
\[ \sum_{j=1}^{N} X_{ij} \leq N Y_i \quad \forall i \quad \text{is "tight"}, \]
i.e., \[ Y_i = \frac{1}{N} \sum_{j=1}^{N} X_{ij} \]

Eliminate \( Y_i \)

Minimize
\[
\sum_{i=1}^{M} \sum_{j=1}^{N} C_{ij} X_{ij} + \sum_{i=1}^{M} \frac{1}{N} F_i \sum_{j=1}^{N} X_{ij}
\]

\[ \Rightarrow \]

Minimize
\[
\sum_{i=1}^{M} \sum_{j=1}^{N} \left[ C_{ij} + \frac{F_i}{N} \right] X_{ij}
\]

s.t.
\[ \sum_{i=1}^{M} X_{ij} = 1 \quad \forall j = 1, \ldots, N \]
\[ X_{ij} \geq 0 \quad \forall i \& j \]

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The solution is  
\[ x^*_i = \begin{cases} 
1 & \text{if } C_{ij} + \frac{F_i}{N} \leq C_{kj} + \frac{F_k}{N} \forall i \\
0 & \text{otherwise} 
\end{cases} \]

with objective value  
\[ \sum_{j=1}^{N} \min_i \left( C_{ij} + \frac{F_i}{N} \right) \]

Although not a strong bound, this is easily computed:

\[ \text{APL} +/ \ L \neq C + \emptyset(\Phi \rho C) \rho F \div N \]
4 = M = # potential plant sites
8 = N = # demand points

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Weak LP Relaxation of Simple Plant Location Problem

The Matrix $C + (F/N)$

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The LP bound is found by summing the minima in each column.

Lower bound provided by weak LP relaxation = 1031.38

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Model #3

Minimize \( \sum_{i=1}^{M} f_i(X_{i1}, X_{i2}, \ldots X_{iN}) \)

subject to \( \sum_{i=1}^{M} X_{ij} = 1 \quad \forall j=1,2,\ldots,N \)

\( X_{ij} \geq 0 \quad \forall i \& j \)

where

\[
f_i(X_{i1}, X_{i2}, \ldots X_{iN}) = \begin{cases} 
0 & \text{if } \sum_{j=1}^{N} X_{ij} = 0 \\
F_i + \sum_{j=1}^{N} C_{ij} X_{ij} & \text{otherwise}
\end{cases}
\]

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Surrogate Constraint

Define a *surrogate multiplier* for each constraint: \( U_j, j=1, \ldots, N; \sum_j U_j = 1 \).

Form a linear combination of the constraints

\[
\begin{align*}
U_1 \times \sum_i X_{i1} &= U_1 \times 1 \\
\vdots \\
U_N \times \sum_i X_{iN} &= U_N \times 1
\end{align*}
\]

\[
\Rightarrow \sum_j U_j \sum_i X_{ij} = \sum_j U_j \Rightarrow \sum_j \sum_i U_j X_{ij} = 1
\]

This *surrogate constraint* is implied by the original set of constraints, but is less restrictive.

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Surrogate Relaxation

We replace the original constraints of Model #3 with the single surrogate constraint:

Minimize \[ \sum_{i=1}^{M} f_i(X_{i1}, X_{i2}, \ldots X_{iN}) \]

subject to \[ \sum_{j} \sum_{i} U_j X_{ij} = 1 \]

\[ X_{ij} \geq 0 \ \forall \ i \& j \]

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Because the objective function is concave, the theory of nonlinear programming assures us that an extreme point of the feasible region (i.e., a basic solution) is optimal, so only a single variable is $\neq 0$.

For example,  

$$X_{ij} = \begin{cases} 
\frac{1}{U_q} & \text{if } i=p, j=q \\
0 & \text{otherwise}
\end{cases}$$

with cost  

$$F_p + C_{pq} \times \frac{1}{U_q}$$

for some $p$ and $q$.  

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Therefore, we can solve the surrogate relaxation by enumerating the MxN basic solutions, and selecting the least cost solution:

\[ S(U) = \min_{i,j} \left\{ F_i + \frac{C_{ij}}{U_j} \right\} \]

Because the optimal solution of the original SPL problem is feasible in this surrogate relaxation,

\[ S(U) \leq \text{optimum of SPL problem} \]

for all \( U = (U_1, U_2, \ldots, U_N) \)
Surrogate Dual Problem

Since for each $U$, $S(U)$ gives us a lower bound on the SPL optimal value, select the surrogate multipliers $U$ to give us the "best", i.e., greatest lower bound:

$$\hat{S} = \text{maximum } S(U)$$

s.t. $\sum_j U_j = 1$
Use of Surrogate Dual bound in a Branch-&-Bound algorithm

Given a value $V$ (e.g., the incumbent solution), we can fathom a subproblem if its surrogate dual value $\hat{S}$ exceeds $V$, and this may be tested without explicitly computing $\hat{S}$:

$$\hat{S} \geq V \iff \exists U=(U_1, \ldots, U_M) \text{ such that } \begin{cases} V \leq F_i + \frac{C_{ij}}{U_j} & \forall i \neq j \\ \sum_j U_j = 1 \end{cases}$$
Assuming $F_i < V$, this is equivalent to

\[
\begin{align*}
U_j & \leq \frac{C_{ij}}{V - F_i} \quad \forall i \& j \\
\sum_j U_j &= 1
\end{align*}
\]

which clearly has a solution if and only if the least upper bounds of $U_j, j=1,...,N$, have a sum $\geq 1$:

\[
\tilde{S} \geq V \iff \sum_j \min_i \left( \frac{C_{ij}}{V - F_i} \right) \geq 1
\]
\[
\frac{C_{ij}}{V - F_i}
\]

\[
\begin{array}{cccccccccc}
0.44 & 0.08081 & 0.06285 & 0.3333 & 0.275 & 0.1481 & 0.2929 & 0 \\
1.076 & 0.06586 & 0.07684 & 0 & 0.4303 & 0.3622 & 0.8595 & 0.7706 \\
0.3443 & 0.07026 & 0.05738 & 0.3478 & 0.2295 & 0.1932 & 0.2037 & 0.274 \\
0.8682 & 0.07973 & 0.03101 & 0.2558 & 0.2713 & 0.3654 & 0.7708 & 0.691 \\
\end{array}
\]

Sum:

\[
\sum_j \min_i \left\{ \frac{C_{ij}}{V - F_i} \right\} = 1.023
\]

The conclusion of the comparison test is:

\[ \hat{S} \geq V \ (= 1031) \]
By any of several methods, the equation

\[ \sum_j \min_i \left\{ \frac{C_{ij}}{F_i} \right\} = 1 \]

may easily be solved for \( \hat{S} \) if the actual value of \( \hat{S} \) is necessary.
Surrogate Dual Algorithm

Lower bound = 1074, Upper bound = 1449
Estimated duality gap = 25.09%

Upper bound achieved by $Y = 1\ 1\ 1\ 1$, i.e.,
opening plants 1 2 3 4

(Not guaranteed to be optimal!)

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Surrogate Dual Algorithm

\[
\text{Matrix } C = \mu (\phi_0 C)^{\rho (SD-F)}
\]

\[
\begin{bmatrix}
0.4198 & 0.0771 & 0.05997 & 0.318 & 0.2624 & 0.1414 & 0.2795 & 0 \\
1.027 & 0.0629 & 0.07339 & 0 & 0.411 & 0.346 & 0.8209 & 0.736 \\
0.3278 & 0.0669 & 0.05464 & 0.3312 & 0.2185 & 0.184 & 0.194 & 0.2609 \\
0.8289 & 0.07612 & 0.0296 & 0.2442 & 0.259 & 0.3489 & 0.7358 & 0.6597
\end{bmatrix}
\]

(\[ y[i] = 1 \] if any column minimum, i.e., Lambda, is found in row \# i of the matrix above)

Surrogate multipliers

\[
\begin{array}{ccccccccc}
  j & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\lambda[j] & 0.3278 & 0.0629 & 0.0296 & 0 & 0.2185 & 0.1414 & 0.194 & 0
\end{array}
\]

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Theorem: If \( \mu_{ij} \geq 0 \) and \( \sum_{j=1}^{N} \mu_{ij} \leq F_i \ \forall i \),

then \( \sum_{j=1}^{N} \min_{i} \{C_{ij} + \mu_{ij}\} \) is a lower bound for the Simple Plant Location problem.

Note: If \( \mu_{ij} = \frac{F_i}{N} \ \forall i,j \), this is the lower bound provided by the LP relaxation of model #2! By appropriate choice of \( \mu_{ij} \), it may give us a better lower bound.
**Proof:** SPL model #1 may be written

\[
\Phi = \text{minimum} \sum_{i,j} C_{ij}X_{ij} + \sum_i \left( F_i - \sum_j \mu_{ij} \right) Y_i + \sum_{i,j} \mu_{ij} Y_i
\]

s.t. \( \sum_i X_{ij} = 1, \ X_{ij} \leq Y_i, \ X_{ij} \geq 0, \ Y_i \in \{0,1\} \ \forall i,j \)

\[
\Rightarrow \Phi \geq \sum_{i,j} C_{ij}X_{ij} + \sum_{i,j} \mu_{ij} Y_i \geq \sum_{i,j} C_{ij}X_{ij} + \sum_{i,j} \mu_{ij} X_{ij} = \sum_{i,j} \left( C_{ij} + \mu_{ij} \right) X_{ij}
\]

\[
\Rightarrow \text{minimum} \sum_{i,j} \left( C_{ij} + \mu_{ij} \right) X_{ij}
\]

s.t. \( \sum_i X_{ij} = 1, \ X_{ij} \leq Y_i, \ X_{ij} \geq 0, \ Y_i \in \{0,1\} \ \forall i,j \)

must give us a lower bound for SPL, namely

\[
\sum_{j=1}^N \min_j \{ C_{ij} + \mu_{ij} \}
\]

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The dual problem is, then, to choose the quantities $\mu_{ij}$ so as to obtain the greatest lower bound, i.e.,

\[
\text{Maximize } \sum_{j=1}^{N} \min_{i} \{c_{ij} + \mu_{ij}\}
\]

s.t. \[ \sum_{j} \mu_{ij} \leq F_{i} \forall i \]

$\mu_{ij} \geq 0 \forall i,j$
Maximize \[ \sum_{j=1}^{N} \min_{i} \{ C_{ij} + \mu_{ij} \} \]

s.t. \[ \sum_{j} \mu_{ij} \leq F_i \ \forall \ i \]

\[ \mu_{ij} \geq 0 \ \forall \ i, j \]

---

**The LP equivalent:**

Maximize \[ \sum_{j=1}^{N} Z_j \]

s.t. \[ Z_j \leq C_{ij} + \mu_{ij} \ \forall \ i, j \]

\[ \sum_{i} \mu_{ij} \leq F_i \ \forall \ i \]

\[ \mu_{ij} \geq 0 \ \forall \ i, j \]

---

The dual of this LP is, in fact, the LP relaxation of SPL model #1!

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**Bilde-Krarup-Erlenkotter (BKE) Algorithm**

This algorithm is a dual ascent algorithm for computing good feasible solutions to the dual of the LP relaxation of Model #1.

At each iteration, exactly one $\mu_{ij}$ is adjusted to give an improvement in the lower bound. It terminates when no improvement can be obtained by adjusting a single multiplier.
Step 1: k=1 & Lambda ← 294 60 28 0 196 132 174 0

Step 2a: ε = 98 0 0 0 0 0 0 0
Lambda[1] = 392
ε = 0 0 98 0, LB = 982

Step 2a: ε = 98 0 0 0 0 0 0 0
Lambda[2] = 60
ε = 0 0 98 0, LB = 982

Step 2a: ε = 98 0 21 0 0 0 0 0
Lambda[3] = 49
ε = 0 0 98 21, LB = 1003

Step 2a: ε = 98 0 21 120 0 0 0 0
Lambda[4] = 120
ε = 0 120 98 21, LB = 1123

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Step 2a: $\epsilon = 98 \ 0 \ 21 \ 120 \ 49 \ 0 \ 0 \ 0$
$\Lambda(5) = 245$
$e = 0 \ 120 \ 147 \ 21, \ LB = 1172$

Step 2a: $\epsilon = 98 \ 0 \ 21 \ 120 \ 49 \ 33 \ 0 \ 0$
$\Lambda(6) = 165$
$e = 33 \ 120 \ 147 \ 21, \ LB = 1205$

Step 2a: $\epsilon = 98 \ 0 \ 21 \ 120 \ 49 \ 33 \ 30 \ 0$
$\Lambda(7) = 204$
$e = 33 \ 120 \ 177 \ 21, \ LB = 1235$

Step 2a: $\epsilon = 98 \ 0 \ 21 \ 120 \ 49 \ 33 \ 30 \ 107$
$\Lambda(8) = 107$
$e = 140 \ 120 \ 177 \ 21, \ LB = 1342$

Step 3: do not terminate. Set $k \leftarrow 2$

Step 2a: $\epsilon = 0 \ 0 \ 21 \ 120 \ 49 \ 33 \ 30 \ 107$
$\Lambda(1) = 392$
$e = 140 \ 120 \ 177 \ 21, \ LB = 1342$
Step 2a: ε = 0 0 21 120 49 33 30 107
Lambda[2] = 60
e = 140 120 177 21, LB = 1342

Step 2a: ε = 0 0 0 120 49 33 30 107
Lambda[3] = 49
e = 140 120 177 21, LB = 1342

Step 2a: ε = 0 0 0 0 49 33 30 107
Lambda[4] = 120
e = 140 120 177 21, LB = 1342

Step 2a: ε = 0 0 0 0 0 33 30 107
Lambda[5] = 245
e = 140 120 177 21, LB = 1342

Step 2a: ε = 0 0 0 0 0 0 30 107
Lambda[6] = 165
e = 140 120 177 21, LB = 1342
Step 2a: \( \epsilon = 00000000107 \)
\( \text{Lambda[7]} = 204 \)
\( \epsilon = 14012017721, \text{LB} = 1342 \)

Step 2a: \( \epsilon = 000000000 \)
\( \text{Lambda[8]} = 107 \)
\( \epsilon = 14012017721, \text{LB} = 1342 \)

Lower bound = 1342, Upper bound = 1342
Duality gap = 0%
No Duality Gap!

Upper bound achieved by \( Y = 1110 \),
i.e., opening plants 1 2 3

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Summary of Results for Example Problem

Optimal Solution of SPL = 1342
LP Relaxation of Model #1 = 1342 0%
Surrogate Relaxation of Model #3 = 1074 20%
LP Relaxation of Model #2 = 1031.38 23%

gap