The Shortest Path Problem

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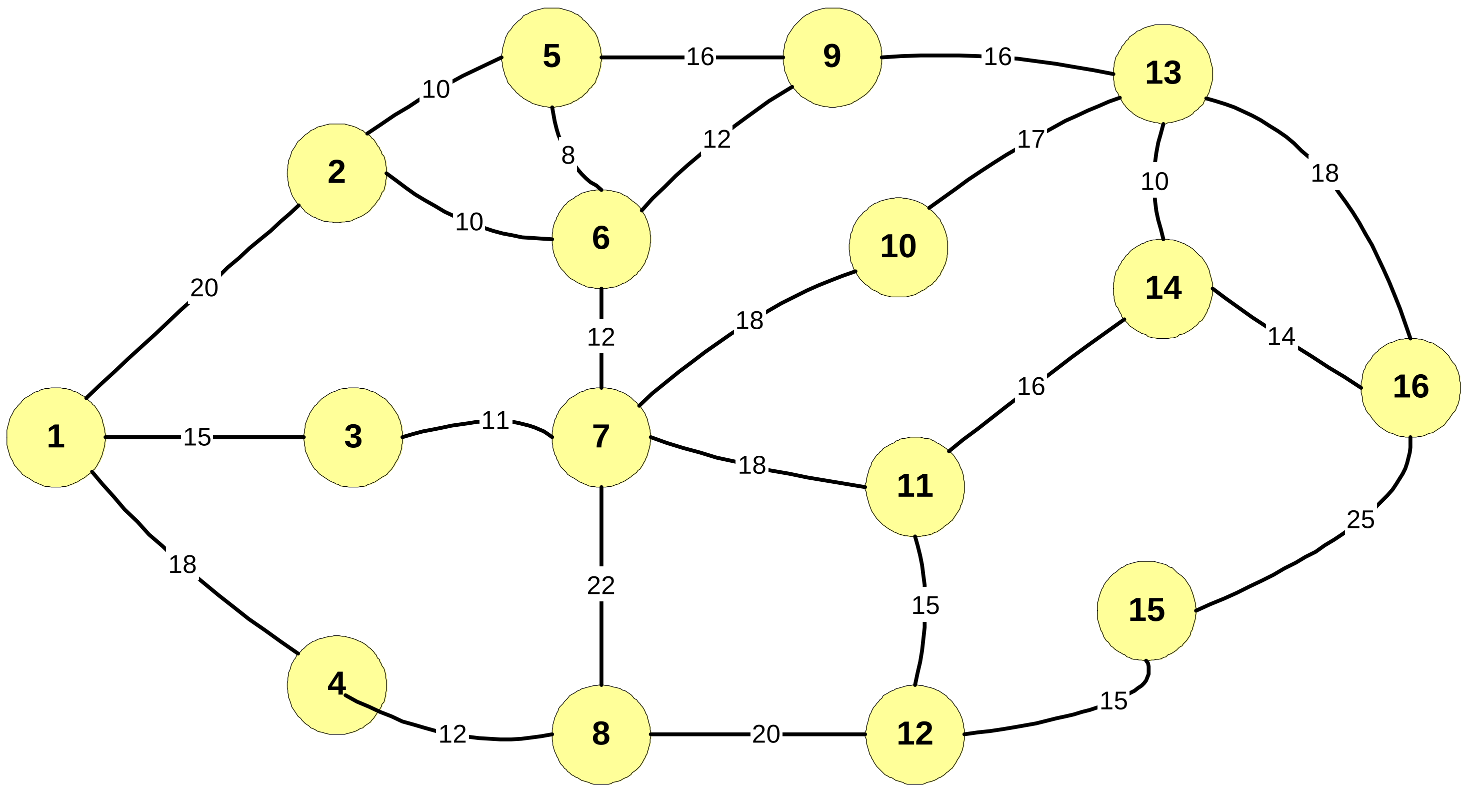


Map of road network between German towns near the Black Forest, where the distances on the links represent kilometers.

What is the shortest route between city #1 and city #16?

Dijkstra's Shortest Path Method ***(a labeling method)***

Finds shortest paths from specified node ***s*** to all other nodes.

Each node ***i*** will have **two labels**:

- a *value V(i)* and
- a *predecessor P(i)*.

Furthermore, at each iteration of the algorithm the labels are categorized as ***temporary*** or ***permanent***.

The value label will record the length of a shortest path from the source node to node ***i*** ***passing through only permanently-labeled nodes.***

The predecessor label will record the (immediate) predecessor of ***i*** on that path.

At each iteration, some temporary labels are updated, and one additional label is made permanent.

When all labels are permanent, the algorithm terminates.

Dijkstra's Shortest Path Method

Let node ***s*** be the source node, and d_{ij} the distance between nodes ***i*** & ***j***.

Step 0:
a. Set ***V(s)***=0 and ***V(j)***=+∞ for all ***j***„ ***s***.
b. Set ***P(i)*** = Φ (null) for all ***i*** (including ***i=s***).
c. Mark labels of node s as permanent.

Step 1:
If all nodes have permanent labels, STOP.
Otherwise, proceed to step 2.

Step 2:
a. Let ***i*** = node whose labels were most recently made permanent.
b. For every temporarily-labeled neighbor ***j*** of node ***i***, update the temporary labels by

$$V(j) \leftarrow \min\{V(j), V(i) + d_{ij}\}$$

c. If label ***V(j)*** was changed, then revise ***P(j)=i***.

Step 3:
a. Find temporarily-labeled node ***j**** having the smallest value label ***V(j)***.
b. Make the labels of node ***j**** permanent.
c. Return to Step 1.

Recovering the Optimal Path

To find the shortest path from node ***s*** to node ***t*** :
Notation: Π will denote the sequence of nodes on the shortest path.

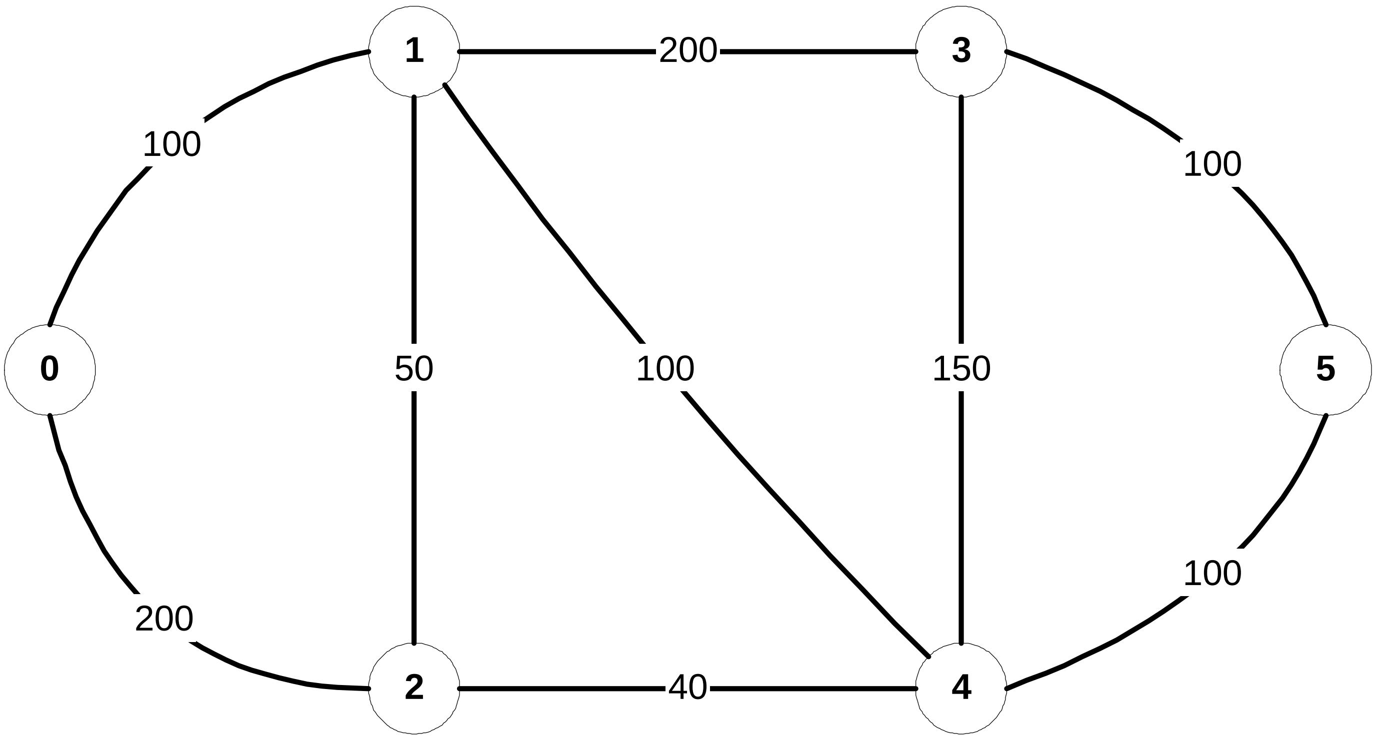
Step 0: Initialize Π = { ***t*** }.

Step 1: Let j = the first entry of the sequence Π.

Step 2: If P(j) $\neq \Phi$, then add j to the beginning of the sequence P.
Otherwise, STOP.

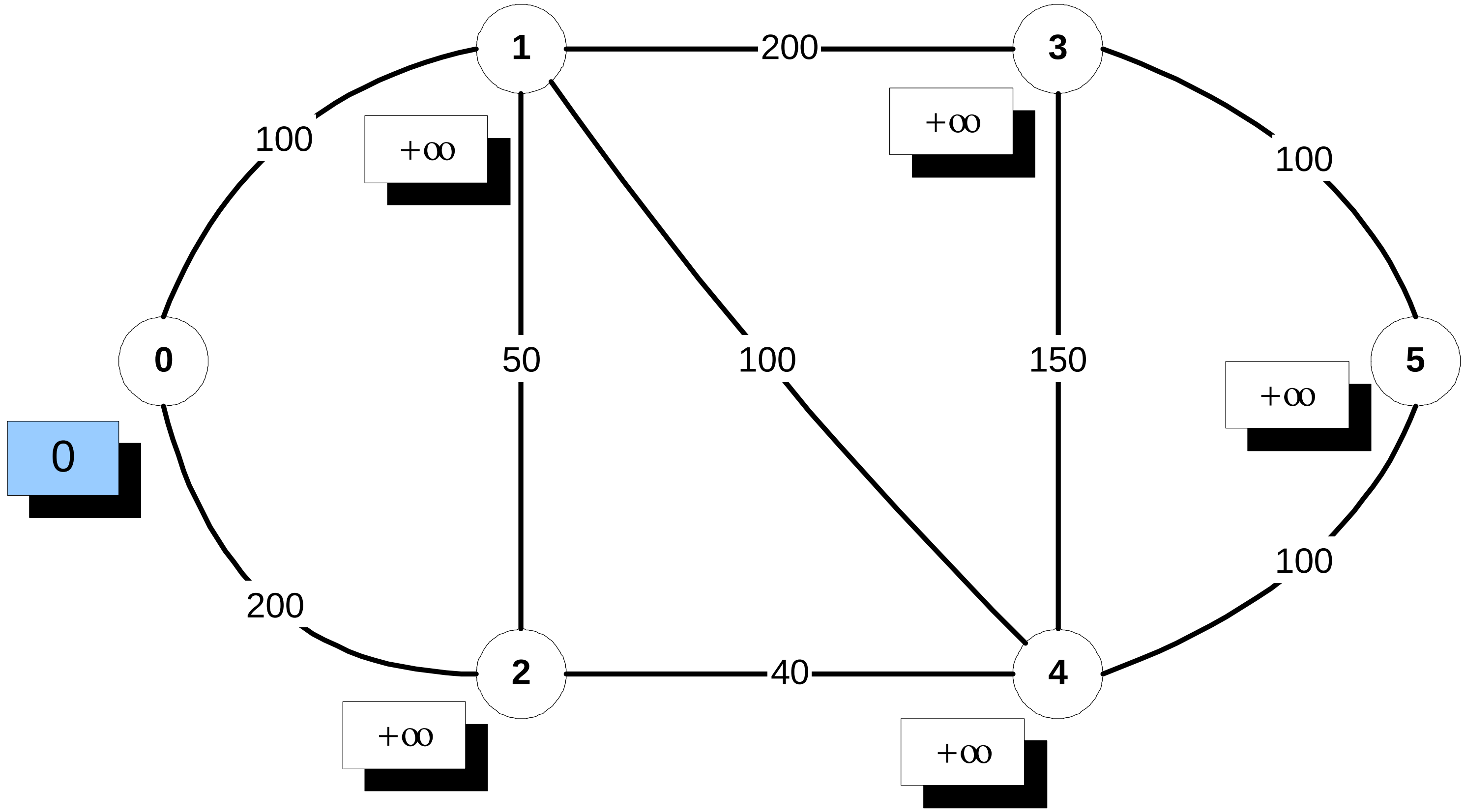
Step 3: Return to step 1.

Example: **Find the shortest path from node 0 to node 5**



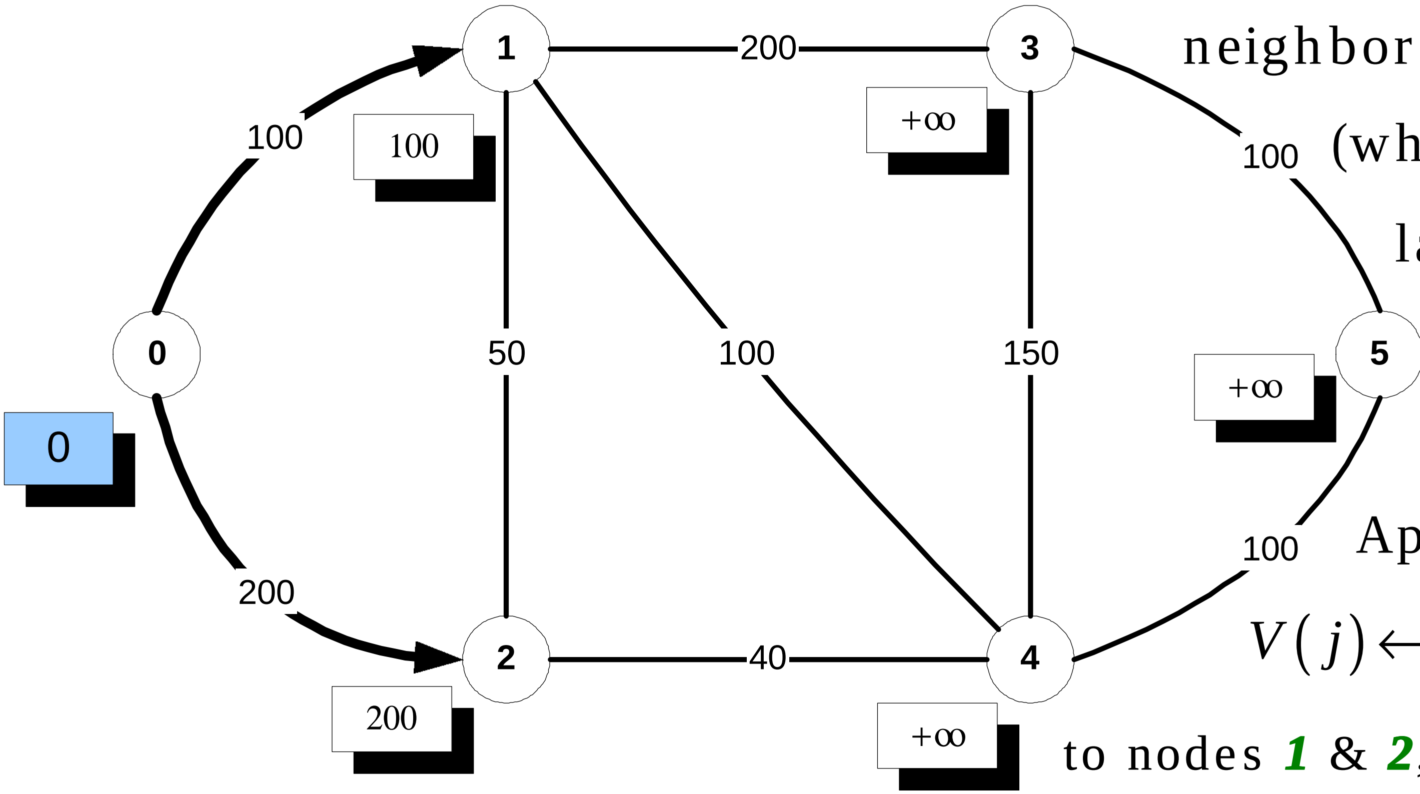
The **value label** will be displayed in a square alongside the node, and the **predecessor label** will be indicated by an arrow entering the node. A label will be indicated to be permanent by shading it.

Step 0. We assign the nodes initially as shown, and make the label of node 0 permanent.



Step 1. Not all nodes have permanent labels, so we proceed to step 2.

Step 2. Node ***0*** label was most recently made permanent, and its neighbors are nodes ***1*** & ***2*** (who now have value labels both +∞.)

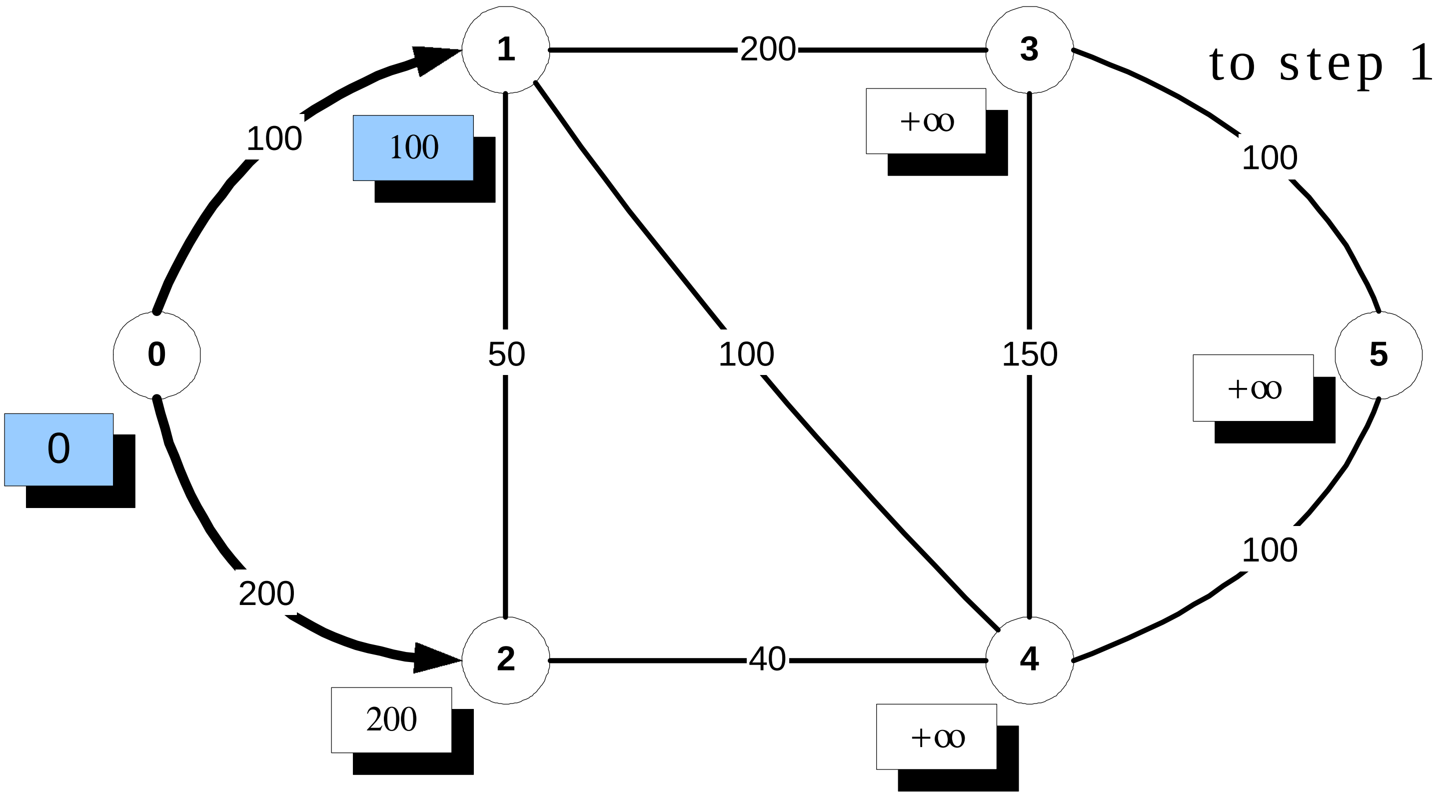


Applying the formula

$$V(j) \leftarrow \min\{V(j), V(i) + d_{ij}\}$$

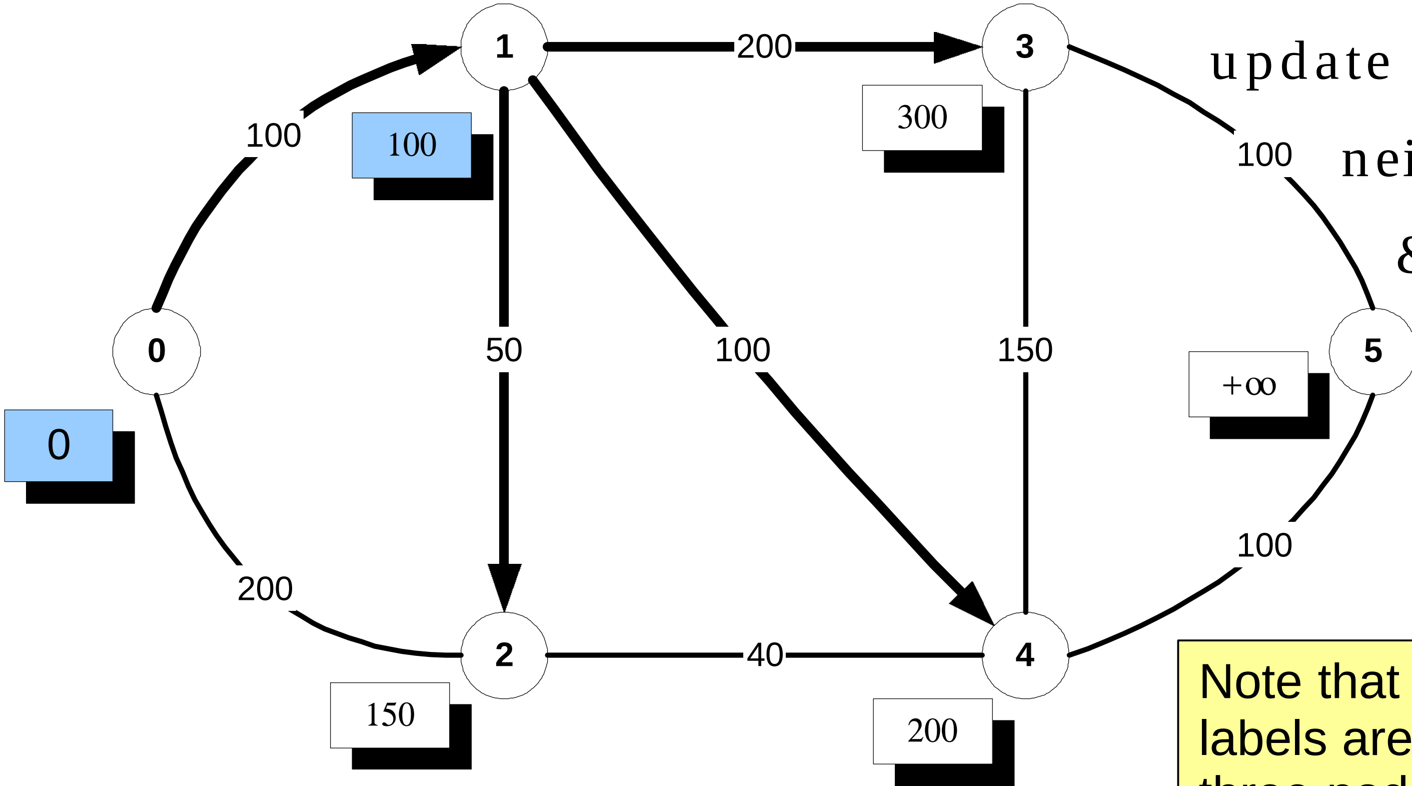
to nodes ***1*** & ***2***, we obtain new values ***V(1)*** = 100, and ***V(2)*** = 200. The predecessor labels are therefore updated, which we are indicating by the arrows, i.e., ***P(1)*** = 0 and ***P(2)*** = 0.

Step 2. The smallest temporary value label is that of node 1, i.e., V(1)=100. Therefore we make the label permanent and return to step 1.



Step 1. Not all labels are permanent (in particular, the labels of nodes 2, 3, & 4, all neighbors of node 1, are temporary). Therefore we proceed to step 2 again.

Step 2. Since node 1 labels were most recently made permanent, we update the labels of its neighbors, nodes 2, 3, & 4:



Note that the predecessor labels are changed for all three nodes.

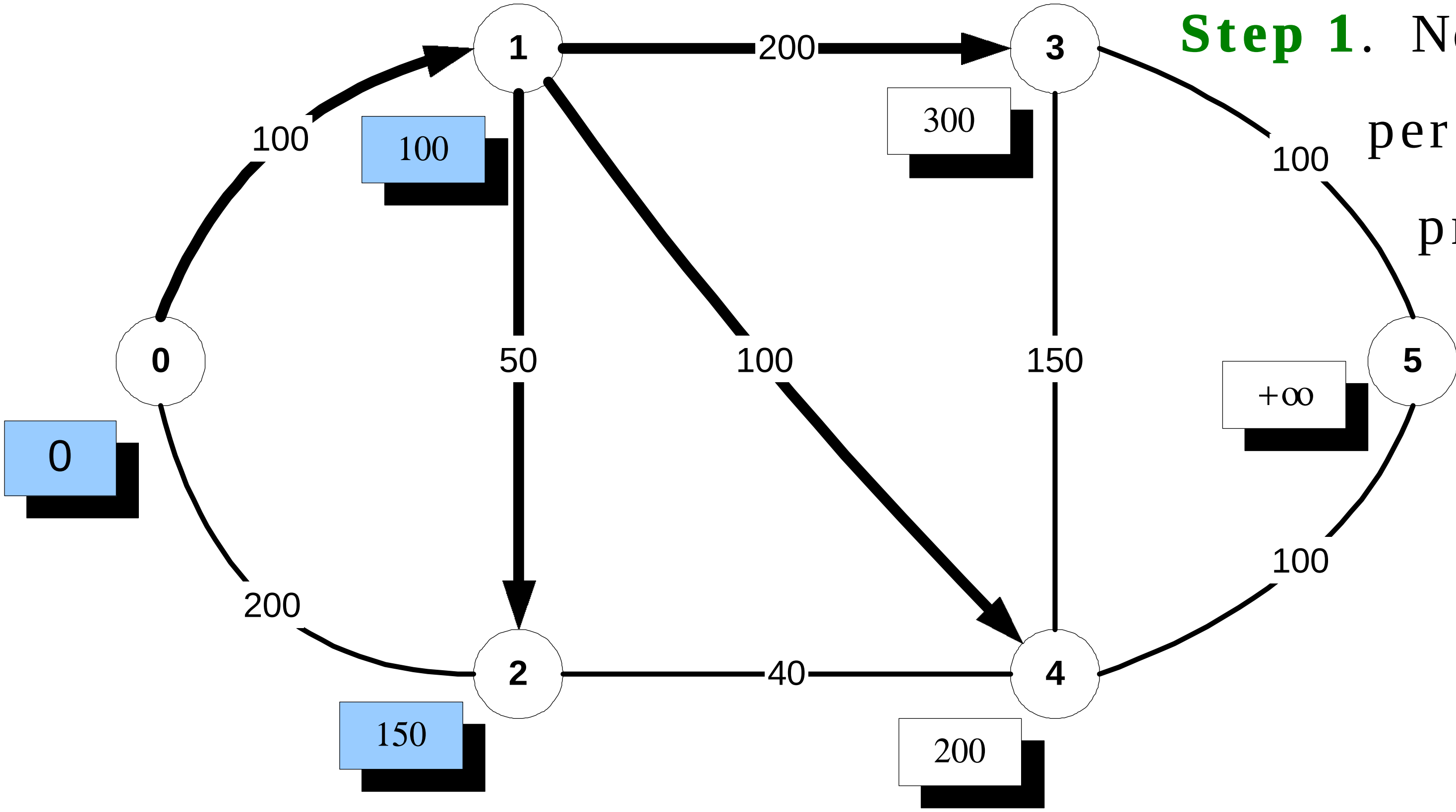
$$V(2) = \min\{200, 100 + 50\} = 150,$$

$$V(3) = \min\{\infty, 100 + 200\} = 300,$$

$$V(4) = \min\{\infty, 100 + 100\} = 200$$

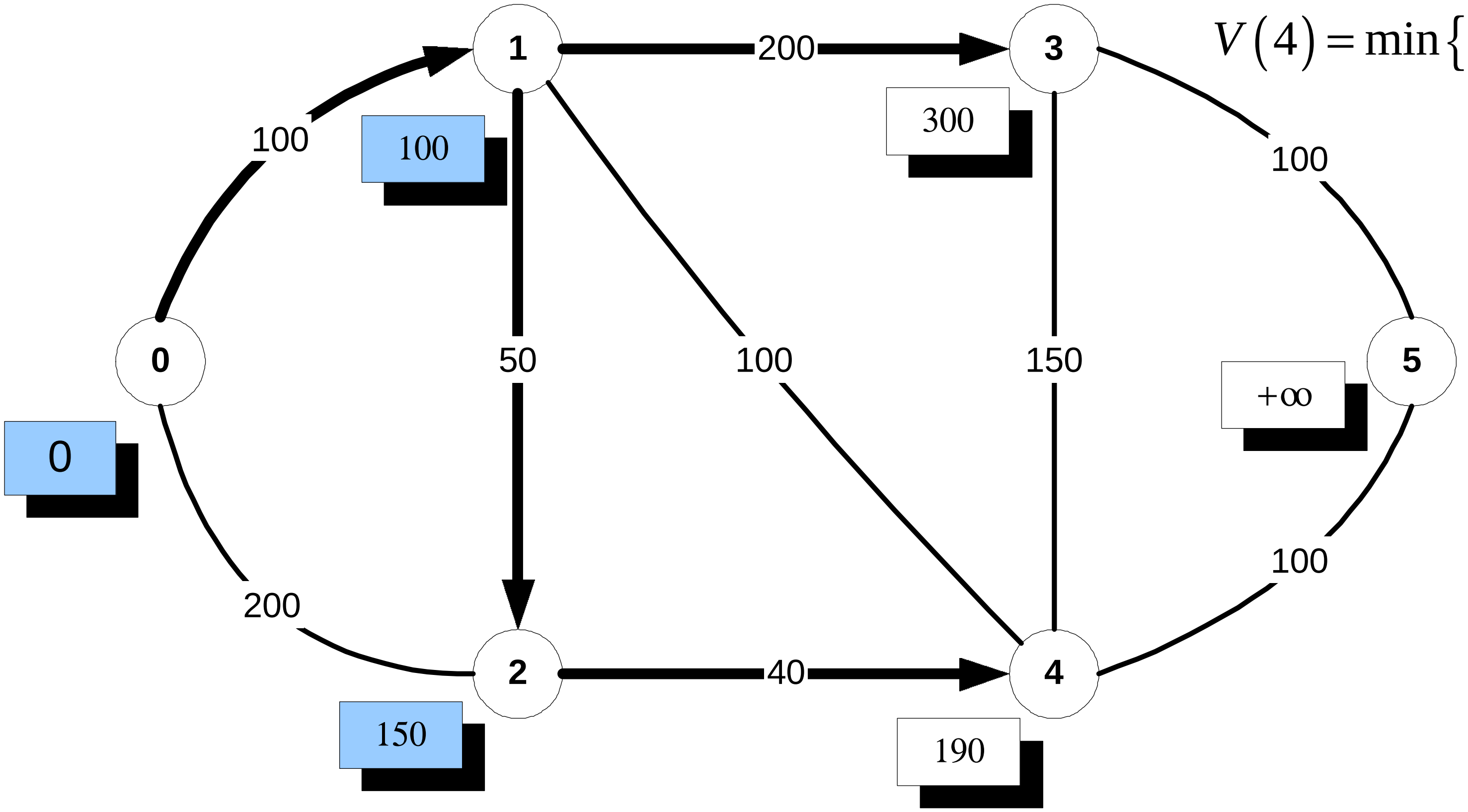
Step 3. The smallest temporary value label is **V(2)=150**, and so we make the **node 2** labels permanent, & return to:

Step 1. Not all labels are permanent, so we proceed to step 2.

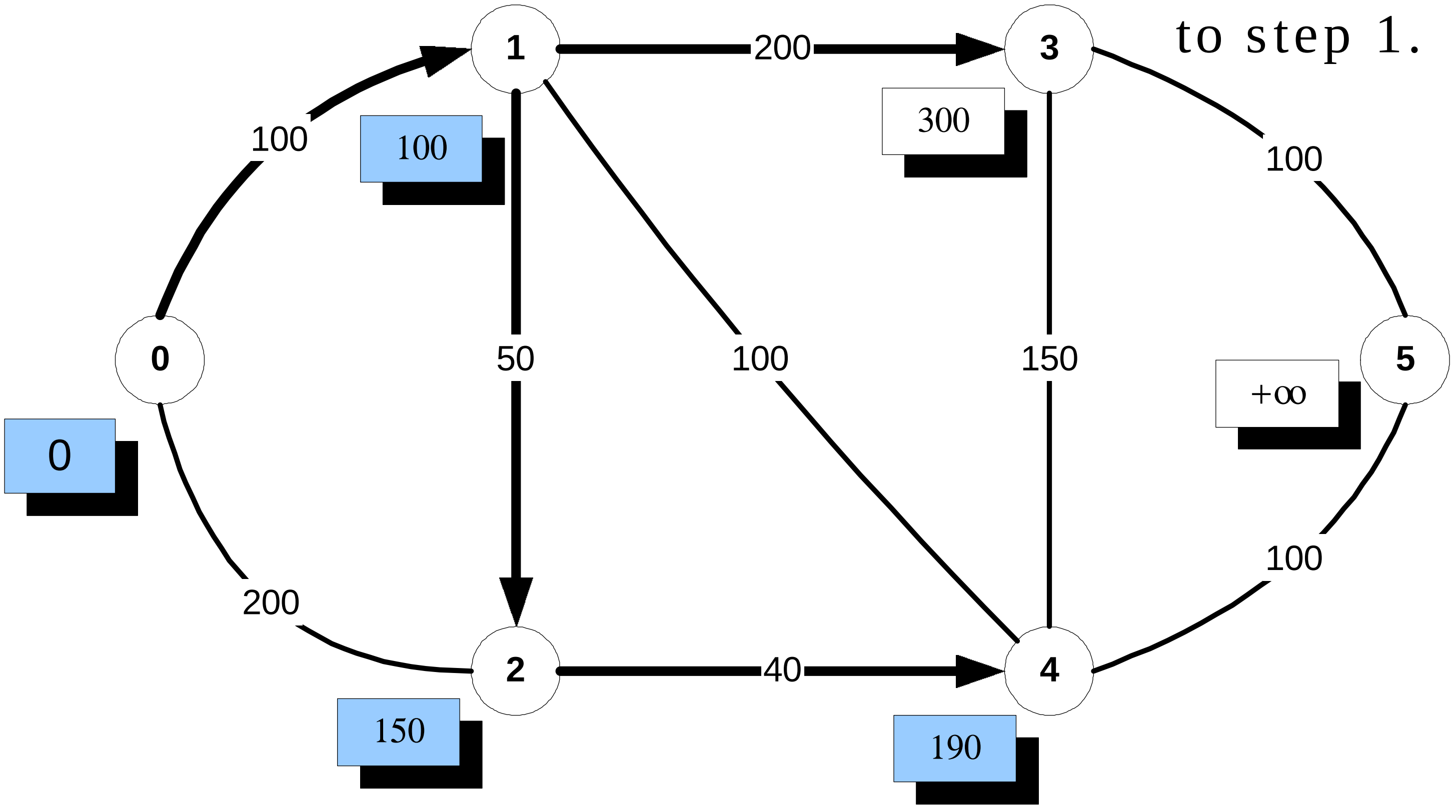


Step 2. Since node **2** labels were most recently made permanent, we update the label of its only unlabeled neighbor, node 4:

$$V(4) = \min\{200, 150 + 40\} = 190$$

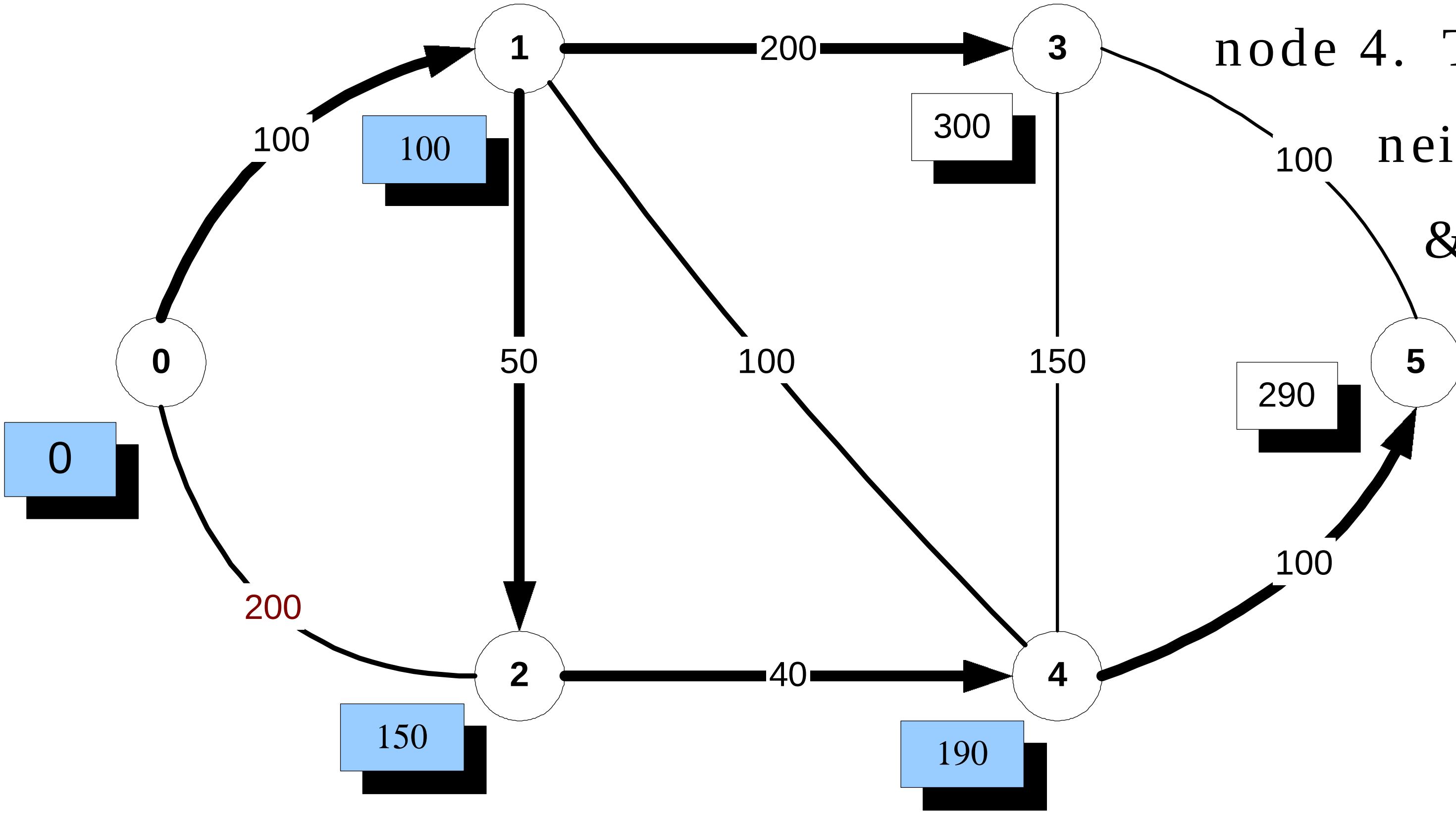


Step 3. The labels of node 4 are next made permanent, and we return to step 1.

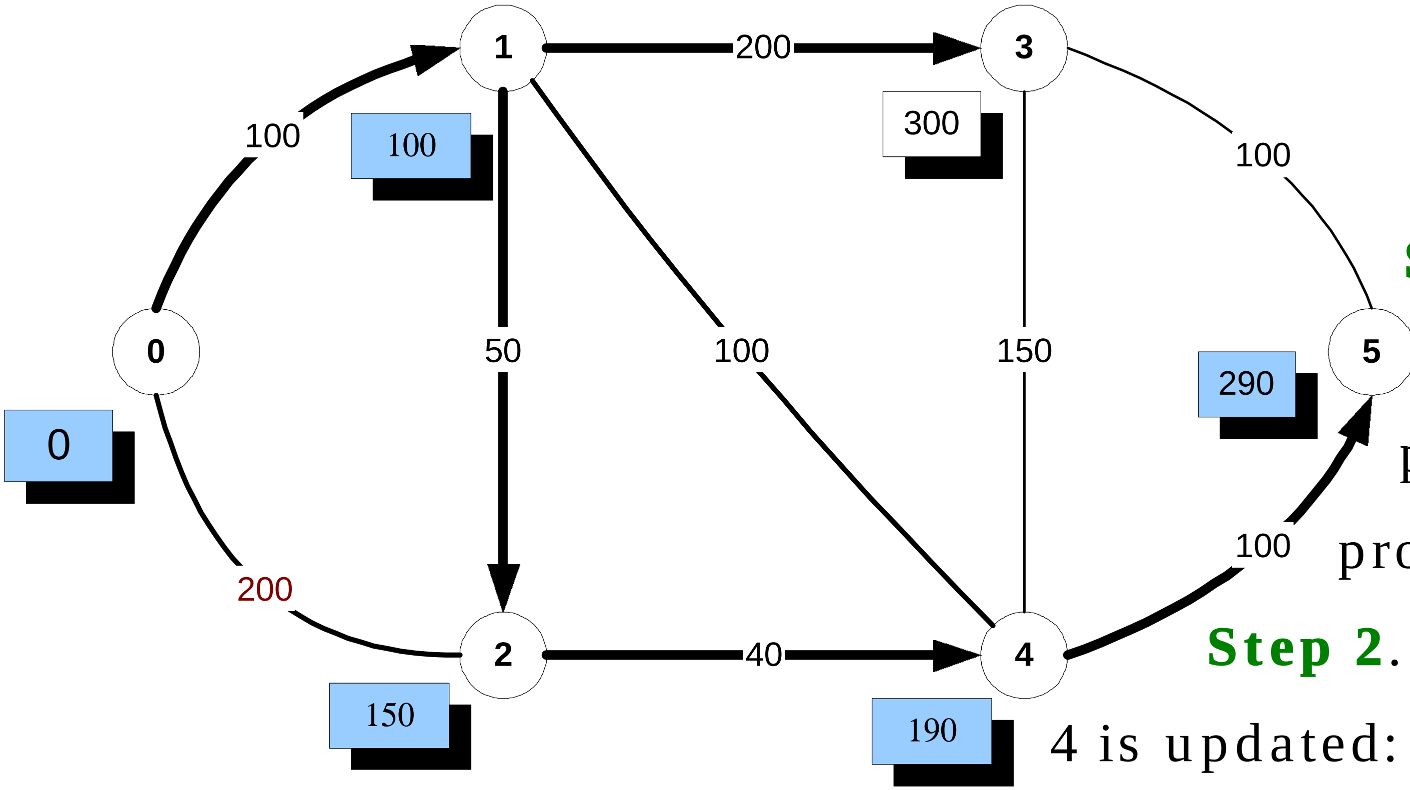


Step 1. Not all labels are permanent, so we proceed to step 2.

Step 2. The node whose labels were most recently made permanent is node 4. The labels of its neighbors, nodes 3 & 5 are next updated.



Step 3. The smallest temporary value label is ***V(5)*** = 290, which is made permanent.



Step 1. Not all labels are permanent, so we proceed to step 2.

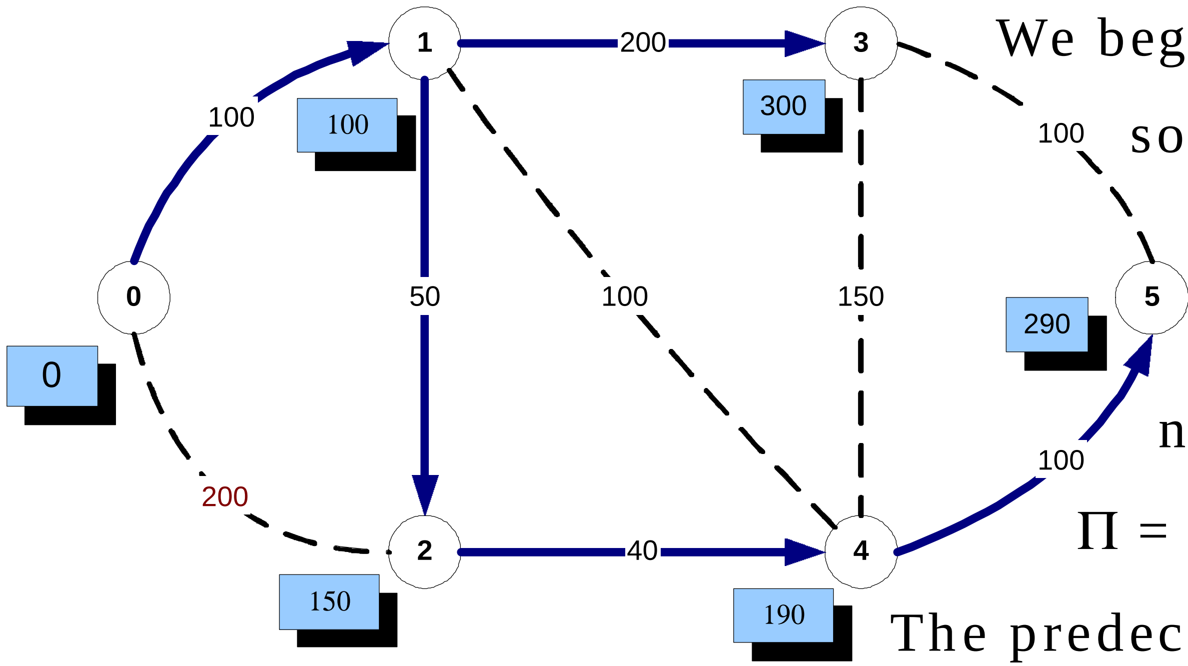
Step 2. The label of node 4 is updated:

$$V(4) = \min\{300, 290 + 100\} = 300$$

Step 3. We make the label of node 4 permanent.

Step 1. All nodes are permanently labeled, and we **STOP**.

Recovering the shortest path from the predecessor labels.



We begin at the ending node, ***5***, so that the sequence is

$\Pi = \{5\}$.

Its predecessor is ***4***, so next

$\Pi = \{4,5\}$.

The predecessor of 4 is ***2***, so $\Pi = \{2,4,5\}$. The predecessor of 2 is ***1***, so $\Pi=\{1,2,4,5\}$. The predecessor of ***1*** is ***0***, so $\Pi=\{0,1,2,4,5\}$. Node ***0*** has <u>no</u> predecessor, and so ***Π is the shortest path***.