Revised Simplex Method

an example...

(includes both Phases I & II)

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$$\text{Minimize } z=3x_1+5x_2+4x_3+7x_4+5x_5+4x_6$$

subject to

$$\begin{cases} 2x_1 - x_2 + x_4 + 3x_6 = 10 \\ x_1 + 3x_3 - x_4 + 3x_5 + 2x_6 \ge 12 \\ 4x_2 + 2x_3 + 3x_4 + x_5 \ge 15 \end{cases}$$

$$\text{and} \quad x_j \ge 0 \quad \forall j = 1, \ldots 6$$

Because of the lack of a slack variable in each constraint, we must use ***Phase I*** *to find an* ***initial feasible basis****.*

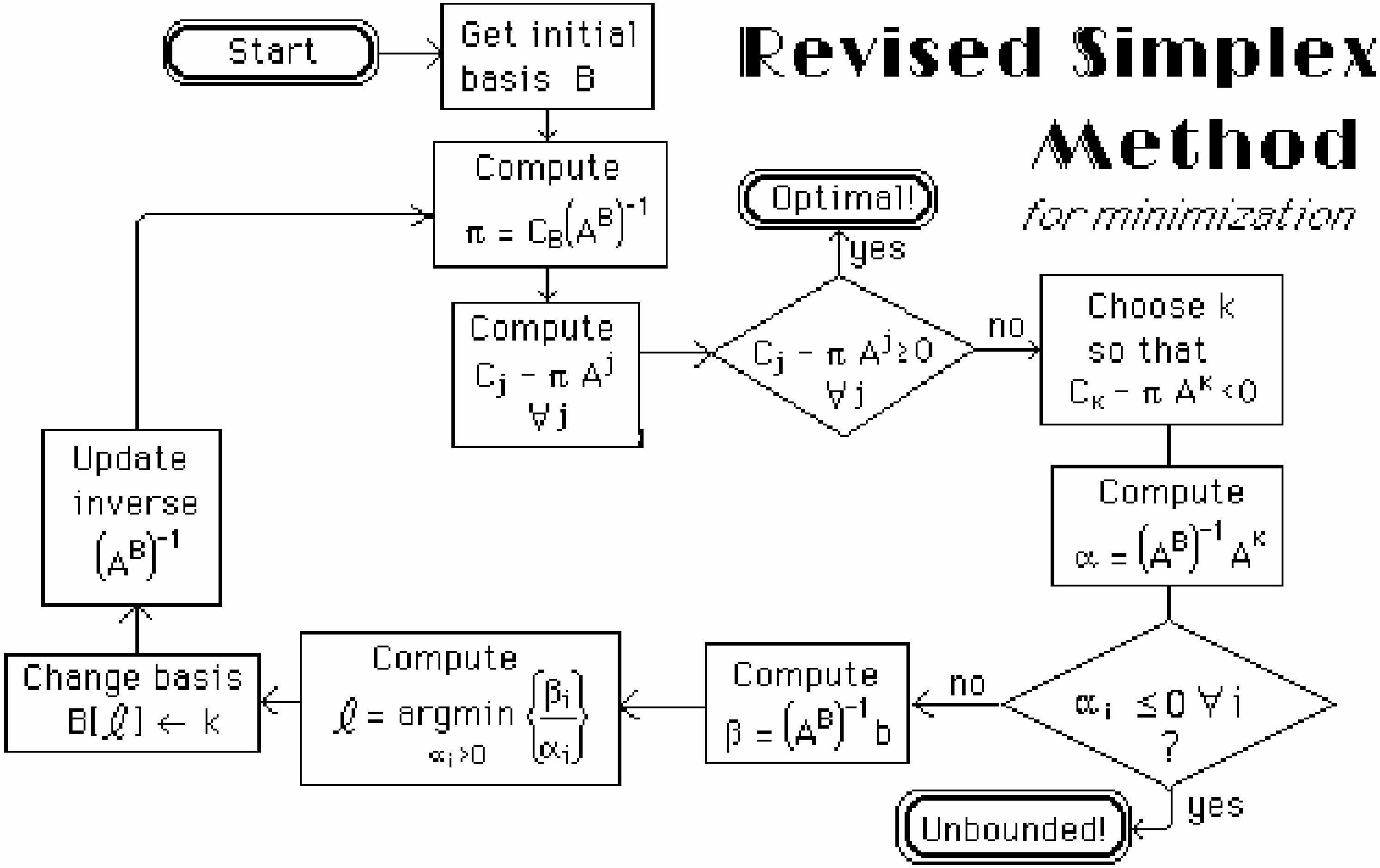
Add variables X_9, X_{10}, X_{11} (artificial variables), and a Phase I ***objective*** *of minimizing the* ***sum*** *of these three variables.*

Phase One

1	2	3	4	5	6	7	8	9	0	1	b	
0	0	0	0	0	0	0	0	1	1	1	0	*phase one objective*
3	5	4	7	5	4	0	0	0	0	0	0	*phase two objective*
2	-1	0	1	0	3	0	0	**1**	0	0	10	
1	0	3	-1	3	2	-1	0	0	**1**	0	12	
0	4	2	3	1	0	0	-1	0	0	**1**	15	

Values of basic (artificial) variables are:

i	Xi
9	10
10	12
11	15


Revised Simplex Method
for minimization
Start
Get initial basis B
Compute
π = C_B(A^B)^-1
Compute
C_j – π A^j
∀ j
C_j – π A^j ≥ 0
∀ j
yes
Optimal!
no
Choose k
so that
C_k – π A^k < 0
Compute
α = (A^B)^-1 A^k
α_i ≤ 0 ∀ i
?
yes
Unbounded!
no
Compute
β = (A^B)^-1 b
Compute
ℓ = argmin {β_i / α_i}
α_i > 0
Change basis
B[ℓ] ← k
Update
inverse
(A^B)^-1

Iteration 1

Current partition: *(**B** = basis, **N** = non-basis)*

B= {9 10 11}, N= {1 2 3 4 5 6 7 8}

Basis inverse is

```
1 0 0
0 1 0
0 0 1
```

$$\Rightarrow \pi = c_B \left(A^B\right)^{-1} = [1,1,1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1,1,1]$$

Simplex multipliers (dual solution):

i	π
1	1
2	1
3	1

Original coefficient matrix A

2	-1	0	1	0	3	0	0	1	0	0
1	0	3	-1	3	2	-1	0	0	1	0
0	4	2	3	1	0	0	-1	0	0	1

$\pi = [1, 1, 1]$

Compute each reduced cost: $c_j - \pi A^j$, e.g.

$\underline{c}_1 = 0 - [1,1,1] \times [2\ 1\ 0] =$ **-3**

Reduced (Artificial) **Costs**

j	Cj	set	
1	-3	N	
2	-3	N	
3	-5	N	←*enter, since* $\underline{C}_j < 0 \Rightarrow$ *decrease in objective!*
4	-3	N	
5	-4	N	
6	-5	N	
7	1	N	
8	1	N	

Any nonbasic variables *except* x_7 & x_8 could be chosen to enter the basis.

```
Entering variable is X[3] from set N
```

Compute the **substitution rates**: $\alpha = \left(A^B\right)^{-1} A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

That is, one unit of x_3 will replace 3 units of the second basic variable (x_{10}) and 2 units of the third basic variable (x_{11}).

Current values of basic variables:

$$x_B = \left[x_9, x_{10}, x_{11}\right] = \left(A^B\right)^{-1} b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 15 \end{bmatrix}$$

Determine variable leaving basis (pivot row):

B	L	X	α	d	ratio	
9	0	10	0	-	inf	
10	0	12	3	↓	4.0	← *minimum ratio*
11	0	15	2	↓	7.5	

α = substitution rate,
d = direction of change of basic variable,
ratio = rhs/α for all α>0

Change of basis

Basic variable **X[10]** leaves basis

New partition: **B**={9 **3** 11}, N={1 2 4 5 6 7 8 **10**}

Updating basis inverse matrix:

affix column of substitution rates alongside the old inverse, and pivot:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \boxed{3} \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 \\ 0 & -\frac{2}{3} & 1 & 0 \end{array}\right]$$

old inverse **new inverse**

$$\Rightarrow \left(A^B\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}$$

Iteration 2

Current partition:

B= {9 3 11}, N= {1 2 4 5 6 7 8 10}

Basis inverse is

1	0	0
0	0.333333	0
0	-0.666667	1

$$\Rightarrow \pi = c_B\left(A^B\right)^{-1} = [1,0,1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} = [1,-\tfrac{2}{3},1]$$

Simplex multipliers (dual solution):

i	π
1	1
2	-0.666667
3	1

Original coefficient matrix A

2	-1	0	1	0	3	0	0	1	0	0
1	0	3	-1	3	2	-1	0	0	1	0
0	4	2	3	1	0	0	-1	0	0	1

Reduced cost of X_1, for example, is $\underline{C}_1$=

$$C_1 - \pi A^1 = 0 - [1, -0.667, 1] \times [2, 1, 0] = -2 + 0.667 = -1.333$$

Reduced (Artificial) **Costs**

j	Cj	set	
1	-1.33333	N	
2	-3	N	
4	-4.66667	N	←*enter*
5	1	N	
6	-1.66667	N	
7	-0.666667	N	
8	1	N	
10	1.66667	N	

any of variables 1, 4, 6, or 7 would improve the phase I objective

Entering variable is X[4] from set N

Compute the ***substitution rates***:

$$\alpha = \left(A^B\right)^{-1} A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{3} \\ \frac{11}{3} \end{bmatrix}$$

Determine variable leaving basis (pivot row):

B	L	X	α	d	ratio	
9	0	10	1.000000	↓	10.00000	
3	0	4	-0.333333	↑	--	
11	0	7	3.666667	↓	1.90909	←*min ratio*

Change of basis

Basic variable X[**11**] leaves basis

New partition: B= {9 3 **4**}, N= {1 2 5 6 7 8 10 **11**}

Updating basis inverse matrix

Write the column α of substitution rates alongside the old inverse matrix, and pivot:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1/3 & 0 & -1/3 \\ 0 & -2/3 & 1 & \boxed{11/3} \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2/11 & -3/11 & 0 \\ 0 & 3/11 & 1/11 & 0 \\ 0 & -2/11 & 3/11 & 1 \end{array}\right]$$

old inverse ***new inverse***

$$\Rightarrow \left(A^B\right)^{-1} = \begin{bmatrix} 1 & 2/11 & -3/11 \\ 0 & 3/11 & 1/11 \\ 0 & -2/11 & 3/11 \end{bmatrix}$$

Iteration 3

Current partition:

B= {9 3 4},   N= {1 2 5 6 7 8 10 11}

Basis inverse is

```
  1  0.181818 -0.272727
  0  0.272727  0.090909
  0 -0.181818  0.272727
```

Simplex multipliers (dual solution): $\pi = c_B \left(A^B\right)^{-1}$

i	π
1	1
2	0.181818
3	-0.272727

$$= [1,0,0]\begin{bmatrix} 1 & 2/11 & -3/11 \\ 0 & 3/11 & 1/11 \\ 0 & -2/11 & 3/11 \end{bmatrix}$$

$$= \left[1, 2/11, -3/11\right]$$

$$\pi = [1,\ 0.181818,\ -0.272727]$$

Original coefficient matrix A

2	-1	0	1	0	3	0	0	1	0	0
1	0	3	-1	3	2	-1	0	0	1	0
0	4	2	3	1	0	0	-1	0	0	1

Reduced cost of X_1, for example, is

$$\underline{c}_1 = c_1 - \pi A^1 = 0 - \begin{bmatrix}1 & 0.181818 & -0.272727\end{bmatrix} \times \begin{bmatrix}2 & 1 & 0\end{bmatrix} = -2.181818$$

Reduced (Artificial) **Costs**

j	Cj	set	
1	-2.18182	N	
2	2.09091	N	
5	-0.272727	N	
6	-3.36364	N	← *enter*
7	0.181818	N	
8	-0.272727	N	
10	0.818182	N	
11	1.27273	N	

any of variables 1, 5, 6, &8 would improve the phase I objective

```
Entering variable is X[6] from set N
```

Compute the ***substitution rates***

$$\alpha = \left(A^B\right)^{-1} A^6 = \begin{bmatrix} 1 & 2/11 & -3/11 \\ 0 & 3/11 & 1/11 \\ 0 & -2/11 & 3/11 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 37/11 \\ 1 \\ -4/11 \end{bmatrix}$$

That is, one unit of x_6 will replace 3.36363 units of the first basic variable (x_9), one unit of the second (x_3), and require 0.36363 additional units of the third (x_4).

```
Determine variable leaving basis (pivot row):
```

B	L	X	α	d	ratio	
9	0	8.09091	3.36363	↓	2.40541	←*min ratio*
3	0	4.63636	0.54545	↓	8.50000	
4	0	1.90909	-0.36363	↑	--	

Change of basis

Basic variable X[9] leaves basis
New partition: B= {6 3 4}, N= {1 2 5 7 8 10 11 9}

All artificial variables (9, 10, 11) are now nonbasic.

*****Feasibility has been achieved! End Phase One*****

Optimal Phase One Solution

i	X[i]	Cz
1	0	0
2	0	0
3	3.32432	0
4	2.78378	0
5	0	0
6	2.40541	0
7	0	0
8	0	0

Phase one objective = sum of artificial variables = 0

Begin Phase II

Iteration 1

```
Current partition:
     B= {6 3 4},   N= {1 2 5 7 8}
```

Basis inverse is

```
        0.297297  0.0540541 -0.0810811
       -0.162162  0.243243   0.135135
        0.108108 -0.162162   0.243243
```

The costs of the basic variables are

$$c_B = [c_6, c_3, c_4] = [4, \quad 4, \quad 7]$$

Compute the simplex multipliers:

$$\pi = c_B\left(A^B\right)^{-1} = [4, \quad 4, \quad 7]\begin{bmatrix} 0.297297 & 0.054054 & -0.810.81 \\ -0.162162 & 0.243243 & 0.135135 \\ 0.108108 & -0.162162 & 0.243243 \end{bmatrix}$$

$$= [1.2973, \quad 0.054054, \quad 1.91892]$$

Simplex multipliers (dual solution)

Original coefficient matrix A

2	-1	0	1	0	3	0	0	1	0	0
1	0	3	-1	3	2	-1	0	0	1	0
0	4	2	3	1	0	0	-1	0	0	1

i	π
1	1.2973
2	0.0540541
3	1.91892

Reduced cost of x_1 is, for example,

$$c_1 - \pi A^1 = 3 - [1.2973, \quad 0.054054, \quad 1.91892] \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0.351351$$

Reduced costs

j	Cj	set	
1	0.351351	N	
2	-1.37838	N	←*enter*
5	2.91892	N	
7	0.054054	N	
8	1.91892	N	

only variable 2 would improve the phase II objective

Entering variable is X[**2**] from set N

Compute the ***substitution rates***:

$$\alpha = \left(A^B\right)^{-1} A^2 = \begin{bmatrix} 0.297297 & 0.054054 & -0.810.81 \\ -0.162162 & 0.243243 & 0.135135 \\ 0.108108 & -0.162162 & 0.243243 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -0.6216 \\ 0.7027 \\ 0.8648 \end{bmatrix}$$

Determine variable leaving basis (pivot row):

B	L	X	α	d	ratio	
6	0	2.40541	-0.6216	↑	--	
3	0	3.32432	0.7027	↓	4.730	
4	0	2.78378	0.8648	↓	3.218	← *min ratio*

Change of basis

Basic variable X[**4**] leaves basis

New partition: B= {6 3 **2**}, N= {1 5 7 8 4}

Update the basis inverse, by writing the substitution rates alongside the old basis inverse, and pivoting:

$$\left[\begin{array}{ccc|c} 0.297297 & 0.054054 & -0.810.81 & -0.6216 \\ -0.162162 & 0.243243 & 0.135135 & 0.7027 \\ 0.108108 & -0.162162 & 0.243243 & \boxed{0.8648} \end{array}\right] \sim \left[\begin{array}{ccc|c} ? & ? & ? & 0 \\ ? & ? & ? & 0 \\ ? & ? & ? & 1 \end{array}\right] \qquad \textit{etc.}$$

When the simplex multipliers and the reduced costs have been computed, the resulting reduced costs are all nonnegative!

Optimal Solution

Objective function: Z= **37.9688**

Primal Solution

(*reduced costs are all nonnegative)*

i	L	X[i]	set	reduced cost
1	0	0	N	0.4375
2	0	3.21875	B	0
3	0	1.0625	B	0
4	0	0	N	1.59375
5	0	0	N	2.53125
6	0	4.40625	B	0
7	0	0	N	0.3125
8	0	0	N	1.53125

Dual Solution

i	π
1	1.125
2	0.3125
3	1.53125