

Example: Revised Simplex Method

Consider the LP:

$$\begin{aligned} &\text{Minimize } z = 3x_1 + 2x_2 + 6x_3 \\ &\text{subject to } \begin{cases} 4x_1 + 8x_2 - x_3 \leq 5 \\ 7x_1 - 2x_2 + 2x_3 \geq 4 \end{cases} \\ &\qquad\qquad\qquad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

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Dept of Mechanical & Industrial Engineering
The University of Iowa

By introducing *slack* and *surplus* variables,
the problem is rewritten with ***equality*** constraints as

$$\text{Minimize } cx \text{ subject to } Ax=b, x \geq 0$$

where

$$c = [3, 2, 6, 0, 0],$$

$$b = [5, 4] \text{ and}$$

$$A = \begin{bmatrix} 4 & 8 & -1 & 1 & 0 \\ 7 & -2 & 2 & 0 & -1 \end{bmatrix}.$$

Although x_4 (the slack variable in 1st constraint) can be used as a basic variable in the first row,
the choice of a basic variable in 2nd constraint is not obvious,
requiring solution of a “Phase One” problem with **artificial** variables introduced.

Suppose that “**Phase One**” has found the **initial basis**
 $B = \{1, 2\}$ for the constraints, i.e., basic variables x_1 and x_2 .

We begin the first iteration of the revised simplex method (RSM)

by computing the **basis inverse matrix**:

$$B = \{1, 2\} \quad \Rightarrow A^B = \begin{bmatrix} 4 & 8 \\ 7 & -2 \end{bmatrix}$$

$$\Rightarrow (A^B)^{-1} = \begin{bmatrix} 0.03125 & 0.125 \\ 0.10937 & -0.0625 \end{bmatrix}$$

Using the basis inverse matrix, we compute the values of the **current basic variables**,

$$\begin{aligned}x_B &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (A^B)^{-1} b \\ &= \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 0.65625 \\ 0.296875 \end{bmatrix}\end{aligned}$$

Next we compute the **simplex multiplier vector** π , to be used in “pricing” the *nonbasic* columns:

$$\begin{aligned}\pi &= c_B (A^B)^{-1} \\ &= [3 \quad 2] \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix} \\ &= [0.3125 \quad 0.25]\end{aligned}$$

Use the simplex multiplier vector π to compute the **reduced cost** of the *nonbasic* variables (x_3, x_4 , & x_5), starting with x_3 :

$$\begin{aligned}\bar{c}_3 &= c_3 - \pi A^3 \\ &= 0 - [0.3125, 0.25] \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= 5.1875 > 0\end{aligned}$$

Since this reduced cost is **POSITIVE**,

increasing x_3 would **increase** the cost.

So x_3 is **rejected** as a pivot variable.

(If we had been maximizing rather than minimizing, of course, then increasing x_3 would benefit the objective!)

We now proceed to the next nonbasic variable, x_4 .

Use the simplex multiplier vector π to compute the reduced cost of the *nonbasic variable* x_4 :

$$\begin{aligned}\bar{c}_4 &= c_4 - \pi A^4 \\ &= 0 - [0.3125, 0.25] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= -0.3125 < 0\end{aligned}$$

Increasing x_4 will improve (i.e. *lower*) the solution, since its reduced cost is **negative!**

Rather than continuing to “price” the remaining nonbasic variables (in this case, only x_5), we will proceed by entering x_4 into the basis! For the *minimum ratio test*, we need the **substitution rates** of x_4 for the basic variables:

$$\begin{aligned}\alpha &= (A^B)^{-1} A^j \\ &= \begin{bmatrix} 0.03125 & 0.125 \\ 0.10937 & -0.0625 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.03125 \\ 0.10937 \end{bmatrix}\end{aligned}$$

That is, one unit of x_4 will substitute for 0.03125 units of the first basic variable and 0.10937 of the second.

Perform the **minimum ratio test** to determine which variable leaves the basis.

$$\min \left\{ \frac{x_B}{\alpha_B} : \alpha_B > 0 \right\} = \min \left\{ \frac{0.6562}{0.0312}, \frac{0.29687}{0.10937} \right\}$$
$$= \min \{21, \mathbf{2.7143}\} = 2.7143$$

Since the **second** ratio is minimum, the second basic variable (i.e., x_2) is replaced by the entering variable x_4 (which will be 2.7143 in the new basic solution), and the **new basis** is $B = \{1, 4\}$.

Update the basis inverse matrix with a **pivot**:

$$\left[\left(A^B \right)^{-1} \mid \alpha \right] = \begin{bmatrix} 0.03125 & 0.125 & 0.03125 \\ 0.10937 & -0.0625 & \boxed{0.109375} \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0.1428 & 0 \\ 1 & -0.5714 & 1 \end{bmatrix}$$

$$\Rightarrow \left(A^B \right)^{-1} = \begin{bmatrix} 0 & 0.1428 \\ 1 & -0.5714 \end{bmatrix}$$

For the new basis $B=\{1,4\}$,

$$c_B = [3,0], A^B = \begin{bmatrix} 4 & 1 \\ 7 & 0 \end{bmatrix}, (A^B)^{-1} = \begin{bmatrix} 0 & 0.142857 \\ 1 & -0.571429 \end{bmatrix}$$

The **basic variables** are $[x_1, x_4] =$

$$x_B = (A^B)^{-1} b = [0.571429 \quad 2.71429]$$

and the new **simplex multipliers** are

$$\pi = c_B (A^B)^{-1} = [0 \quad 0.428571]$$

The *reduced costs* of the nonbasic variables {2, 3, 5} are now:

$$\bar{c}_2 = c_2 - \pi A^2 = 2 - [0 \quad 0.428571] \begin{bmatrix} 8 \\ -2 \end{bmatrix} = 2.85714 > 0$$

$$\bar{c}_3 = c_3 - \pi A^3 = 6 - [0 \quad 0.428571] \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 5.14286 > 0$$

$$\bar{c}_5 = c_5 - \pi A^5 = 0 - [0 \quad 0.428571] \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0.428571 > 0$$

Since the reduced costs are all **positive**,

the current solution $[x_1 \quad x_4] = [0.571429 \quad 2.71429]$

is **optimal!**