Efficiency of the Revised Simplex Method

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Which version
  • "ordinary" simplex method
  • "revised" simplex method
requires the least computational effort?

Computational effort per pivot depends
upon the problem parameters

\[ n = \text{# columns of } A \]
\[ m = \text{# constraints} \]
\[ d = \text{density of } A (\% \text{ nonzero elements}) \]
Assume that, in the ordinary simplex tableau, previous pivots have increased the density such that we cannot make good use of sparse matrix techniques.

Let's count the number of multiplications & divisions per pivot.
Consider the operations in a pivot in row $r$, column $s$:

$$
\begin{array}{cccccc}
\hat{C}^1 & \hat{C}^2 & \ldots & \hat{C}^s & \ldots & \hat{C}^n \\
\hat{A}_1 & \hat{A}_1 & \ldots & \hat{A}_1 & \ldots & \hat{A}_1 \\
\hat{A}_2 & \hat{A}_2 & \ldots & \hat{A}_2 & \ldots & \hat{A}_2 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\hat{A}_r & \hat{A}_r & \ldots & \hat{A}_r & \ldots & \hat{A}_r \\
\hat{A}_m & \hat{A}_m & \ldots & \hat{A}_m & \ldots & \hat{A}_m \\
\end{array}
\begin{array}{c}
\hat{Z} \\
b_1 \\
b_2 \\
\vdots \\
b_r \\
b_m \\
\end{array}
$$
Ordinary Simplex Method
Pivoting in full tableau, with 100% density

Revised Simplex Method
Explicit basis inverse maintained, and density less than 100%

Comparison of Algorithms
Ordinary Simplex Method

Operation Count
\( (x \text{ and } \div) \)
per iteration

- Minimum Ratio Test (pivot row selection)
  \( m \) divisions

**Pivot:**
- Divide row \( r \) by \( \tilde{A}_r^s \) (need not divide in basic columns)
  \( n-m \) divisions
For \( i=1,2,...,m+1, i \neq r \),

\[
\text{add } -\hat{A}_i^g \text{ times row } r \text{ to row } i
\]

(only necessary to compute elements in nonbasic columns)

\((n-m)\) multiplications per each of \( m \) rows
Total number of multiplications & divisions:

\[ N_S = m + (n-m) + m(n-m) \]

\[ = m + n + mn - m^2 \]

per iteration.
Revised Simplex Method

Operation Count
(x and ÷)
per iteration

- Pricing each of (n-m) nonbasic columns
  (selecting pivot column)

\[ \bar{c}^j = \pi A^j \]

(dm) multiplications per each of (n-m) columns
Computing substitution rates \[ \tilde{A}^s = (A^B)^{-1} A^j \]
(computing pivot column)

\[ dm \] multiplications per each of \( m \) rows

Minimum ratio test (pivot row selection)

\[ m \] divisions
Pivot (update of basis inverse matrix, rhs, \( \pi \))

- divide row \( r \) of \((A^B)^{-1} \) & \( \hat{b} \) by pivot element (m+1) divisions

- For \( i = 0, 1, 2, \ldots m \) (i = r):
  Add multiple of row \( r \) to row \( i \)
  (m+1) multiplications per each of \( m \) rows
Revised Simplex Method

Total number of multiplications & divisions:

\[ N_R = dm(n-m) + dm^2 + m + (m+1) + (m+1)n \]

\[ = dm n + m^2 + 3m + 1 \]

per iteration.
Comparison of Algorithms

Multiplications & Divisions per iteration:

Ordinary Simplex \( N_S = m + n + mn - m^2 \)

Revised Simplex \( N_R = dm n + m^2 + 3m + 1 \)

Under what conditions is the revised simplex method more efficient than the ordinary simplex method?

That is, when is \( N_R < N_S \)?
\[ N_R < N_S \]

\[ \Rightarrow dm n + m^2 + 3m + 1 < m + n + mn - m^2 \]

\[ \Rightarrow dm n < mn + n - 2m^2 - 2m - 1 \]

\[ \Rightarrow d < 1 - 2 \frac{m}{n} + \frac{1}{m} - \frac{2}{n} - \frac{1}{mn} \approx 1 - 2 \frac{m}{n} \]

\( \text{negligible} \)
So the revised simplex method is more efficient than the ordinary simplex method when the density of the coefficient matrix $A$ satisfies:

$$d < 1 - 2\frac{m}{n}$$

For example:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$1 - 2\frac{m}{n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>50</td>
<td>60%</td>
</tr>
<tr>
<td>100</td>
<td>1000</td>
<td>80%</td>
</tr>
<tr>
<td>100</td>
<td>10000</td>
<td>98%</td>
</tr>
</tbody>
</table>

If $m=10$ & $n=50$, then the revised simplex method is more efficient if the density is less than about 60%.
\[ N_S = m + n + mn - m^2 \]
\[ N_R = dmn + m^2 + 3m + 1 \]

For large LP problems in the "real world", the density is typically no more than 5%.

If \( m=100 \) and \( n=1000 \), \( N_S = 91100 \)

\[
\begin{array}{|c|c|c|}
\hline
 & \text{d=1\%} & \text{d=5\%} \\
\hline
N_R & 11301 & 15301 \\
\hline
N_R/N_S & 0.124 & 0.168 \\
\hline
\end{array}
\]