Relaxations

in Mathematical Programming

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Consider a constrained optimization problem

$$\textbf{P:} \quad z = \min\left\{c(x) \mid x \in X \subseteq R^n\right\}$$

and a problem

$$\textbf{P}^{\text{R}}\textbf{:} \quad z^R = \min\left\{f(x) \mid x \in T \subseteq R^n\right\}$$

The problem **P**$^{\text{R}}$ is a **relaxation** of problem **P** if:

- $X \subseteq T$, i.e., every x feasible in P is also feasible in P$^{\text{R}}$,

and

- $f(x) \leq c(x) \quad \forall x \in X$

Proposition: If **P**$^{\text{R}}$ is a relaxation of **P**, then its optimal value is a *lower* bound of the optimal value of **P**:

$$z^R \leq z.$$

Notes:

- The solution z^R of relaxation **P**^{R} provides a guaranteed estimate on the quality of a proposed solution of **P**: for any feasible $x \in X$, the maximum relative error is

$$\frac{c(x) - z^R}{z^R}.$$

- Relaxations are most frequently used in branch-&-bound algorithms for combinatorial problems (providing a bound used in "fathoming" nodes of the search tree.)

- To be useful, **P**^{R} must be more easily solved than **P.**

- If P is a *maximization* problem, then the second condition in the definition of a relaxation is $f(x) \geq c(x) \quad \forall x \in X$ and as a result, the relaxation provides an *upper* bound on z.

Linear Programming Relaxation of Integer & Mixed-Integer LP

The most common relaxation of IP problems is the **LP relaxation**, in which integer restrictions are removed:

$$P: \quad z = \min\left\{cx \mid Ax \geq b, x \in Z_+^n\right\}$$

where Z_+^n is the set of n-dimensional vectors of non-negative integers.

$$P^{LP}: \quad z^{LP} = \min\left\{cx \mid Ax \geq b, x \in R_+^n\right\}$$

Note: in the definition of relaxation, let

$$c(x) = cx = f(x) \text{ and}$$

$$X \equiv \left\{x \mid Ax \geq b, x \in Z_+^n\right\} \,\&\, T \equiv \left\{x \mid Ax \geq b, x \in R_+^n\right\}$$

so that $X \subset T$

I.e., while the objective functions of P & P^{LP} are the same, relaxing the integer restrictions of an IP adds feasible solutions to the problem, so that a lower minimum might be found.

Lagrangian Relaxation of an Integer Programming Problem

Consider the IP problem

$$P: \quad z = \min\left\{cx \mid Ax \geq b, x \in X \subseteq Z_+^n\right\}$$

Often, X is defined by additional linear constraints on the integer variables, i.e., $X = \left\{x \mid Dx \geq e, x \in Z_+^n\right\}$.

Dropping the constraints $Ax \geq b$ obviously satisfies the definition of a relaxation, since

- the first condition is satisfied (the feasible region is expanded)
- the second condition is trivially satisfied (the objective is unchanged).

To obtain a more useful relaxation, we change the objective function as well, using a vector of ***Lagrangian multipliers***.

Suppose that A is $m \times n$, i.e., m constraints are being relaxed.

Let $\lambda \in R_+^m$ be a vector of nonnegative numbers (*Lagrangian multipliers*),

one for each relaxed constraint.

For example, λ_i is the multiplier for constraint i:

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad \text{i.e.,} \quad \sum_{j=1}^{n} a_{ij} x_j - b_i \geq 0.$$

In the feasible region, then, the product of λ_i and $\sum_{j=1}^{n} a_{ij} x_j - b_i$ is non-

negative, i.e., $\lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) \geq 0$, so that

$$\sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) \leq \sum_{j=1}^{n} c_j x_j$$

The ***Lagrangian relaxation*** of P is therefore defined to be

$$P^L(\lambda): \quad z^L(\lambda) = \min\left\{cx - \lambda\left(Ax - b\right) \mid x \in X\right\}$$

since, as we have shown,

for any $\lambda \geq 0$ and $x \in X$, $f(x) \equiv cx - \lambda\left(Ax - b\right) \leq cx$

Note that outside the feasible region,

$$Ax - b \leq 0 \Rightarrow \lambda\left(Ax - b\right) \leq 0$$

so that the objective $f(x) \equiv cx - \lambda\left(Ax - b\right)$ may be thought of as including a *penalty for violating the constraints.*

Lagrangian Duality

Every choice of the Lagrangian multipliers $\lambda \geq 0$ yields a Lagrangian relaxation, i.e., a lower bound on the optimal value z.

The ***Lagrangian dual problem*** is to choose multipliers to obtain the greatest lower bound, i.e.,

$$D^L: \quad \hat{z}^L = \max\left\{z^L(\lambda) \mid \lambda \geq 0\right\}$$

This is, in effect, a ***maxi-min*** problem, since evaluating the dual objective function $z^L(\lambda)$ requires solving a minimization problem.

Note that $\hat{z}^L \leq z$, i.e., $z - \hat{z}^L \geq 0$. This nonnegative difference is called the ***duality gap***.

Lagrangian Dual Problem:

Find $\lambda \geq 0$ so that the Lagrangian relaxation yields the *greatest lower bound* of z:

$$\hat{P}^L : \quad \hat{z}^L = \max\left\{ z^L(\lambda) \mid \lambda \geq 0 \right\}$$

Obviously,

$$z \geq \hat{z}^L \geq z^L(\lambda) \qquad \forall \lambda \geq 0$$

and the difference $z - \hat{z}^L \geq 0$ is called the Lagrangian ***duality gap***.

If λ^* is the optimal dual solution, then the solution $x(\lambda^*)$ of the Lagrangian relaxation $P^L(\lambda^*)$ is generally *infeasible* in the primal problem, i.e., $Ax(\lambda^*) \geq b$ is violated.

If $x(\lambda^*)$ is feasible in the primal, is it optimal???

Sometimes $x(\lambda^*)$ can be easily adjusted so as to satisfy the constraints (although optimality is not guaranteed)…

a so-called ***"Lagrangian heuristic"*** method

The constraints of an IP may be partitioned in several ways

$$\begin{bmatrix} A \\ \cdots \\ D \end{bmatrix} x \geq \begin{bmatrix} b \\ \cdots \\ e \end{bmatrix}$$

where $X = \left\{ x \mid Dx \geq e, x \in Z_+^n \right\}$, so that several Lagrangian dual problems may be defined, with duality gaps of various sizes. *(See Generalized Assignment Problem (GAP))*

"No free lunch" principle: usually, the smaller the duality gap, the more difficult it is to solve the Lagrangian relaxation!

Surrogate Duality

As in Lagrangian duality, nonnegative multipliers are defined, but used to *aggregate* the constraints:

$$\mu \geq 0 \,\&\, Ax \geq b \Rightarrow \mu Ax \geq \mu b$$

Surrogate Relaxation:

$$P^S(\mu): \quad z^S(\mu) = \min\{cx \mid \mu Ax \geq \mu b, x \in X\}$$

$P^S(\mu)$ is easily seen to be a relaxation, since

- the objective is unchanged
- the feasible region is enlarged

Surrogate Dual Problem:

$$\hat{P}^S: \quad \max\{\hat{z}^S(\mu) \mid \mu \geq 0\}$$

Lagrangian Duality

Every choice of the Lagrangian multipliers $\lambda \geq 0$ yields a Lagrangian relaxation, i.e., a lower bound on the optimal value z.

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Combinatorial or IP problems may be classified as

- **"Easy" problems**

  polynomial-time algorithms exist

  examples: shortest path problem
  minimum spanning tree problem
  transportation problem
  assignment problem

- **"Hard" problems**

  no polynomial-time algorithms are known

  examples: traveling salesman problem
  scheduling problems
  quadratic & generalized assignment problems

Often a hard problem may be modeled as an easy problem with additional complicating constraints.

Example: ***Generalized Assignment Problem***

a multiple-choice problem, with additional machine capacity limits

Example: ***Shortest Hamiltonian Path Problem*** *(like a traveling salesman problem except route is a path rather than a cycle)*

a minimum spanning tree problem, with node degrees at most 2.