Introduction to Queueing: $M/M/1/N/N$

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dennis-bricker@uiowa.edu
**M/M/1/N/N**

- Single server
- **Finite Source Population of size N**
- Arrival & Service processes are **Memoryless**, i.e., service times have Exponential distribution with mean $\frac{1}{\mu}$
- A departing customer returns to the queue after a time having an Exponential distribution with mean $\frac{1}{\lambda}$
Each customer, after being served, returns to the source population for a length of time having exponential distribution with mean $1/\lambda$. 

©Dennis Bricker, U. of Iowa, 1997
Queueing Intro - Part 4

M/M/1/N/N

Birth/Death Model

\[ \begin{align*}
0 & \rightarrow 1 & 2 & \rightarrow 3 & \cdots & \rightarrow N-1 & \rightarrow N \\
\mu & \rightarrow N\lambda & (N-1)\lambda & (N-2)\lambda & (N-3)\lambda & 2\lambda & \lambda \\
\end{align*} \]
\[ \pi_0 = \frac{1}{\sum_{j=0}^{N} \frac{N!}{(N-j)!} \rho^j \pi_0} \]

First calculate the probability \( \pi_0 \) that the server is idle.

\[ \pi_j = \frac{N!}{(N-j)!} \rho^j \pi_0 \]

Other probabilities are then multiples of \( \pi_0 \)

where \( \rho = \frac{\lambda}{\mu} \)

©Dennis Bricker, U. of Iowa, 1997
Example

An operator can be assigned to service (load, unload, adjust, etc.) several automatic machines in a factory.

- Running time of each machine before it must be serviced has an exponential distribution, with mean 120 minutes.
- Service time has an exponential distribution with mean 12 minutes.

To achieve a desired utilization of ≥ 87.5% for the machines, how many machines should be assigned to the operator?
This can be modeled as a M/M/1 queueing system with finite source population.

Machine operator = server
Machines = customers
\[ \mu = 5/\text{hour} \]
\[ \lambda = 0.5/\text{hour} \]
\[ \frac{1}{\pi_0} = 1 + 3 \rho + 3 \times 2 \times \rho^2 + 3! \times \rho^3 \]

\[ \lambda = \frac{1}{2} \text{hrs} = 0.5/\text{hr} \]
\[ \mu = 5/\text{hr} \]
\[ \rho = \frac{\lambda}{\mu} = 0.1 \]
\[
\frac{1}{\pi_0} = \sum_{j=0}^{3} \frac{3!}{(3-j)!} (0.1)^j
\]

\[
= 1 + 0.3 + 0.06 + 0.006
\]

\[
= 1.366
\]

\[
\pi_0 = \frac{1}{1.366} = 0.732965
\]

\[
\pi_1 = 0.3 \pi_0 = 0.2196
\]

\[
\pi_2 = 0.06 \pi_0 = 0.0439
\]

\[
\pi_3 = 0.006 \pi_0 = 0.0044
\]

Steady-state Distribution

i.e., operator will be idle about 73% of the time!

©Dennis Bricker, U. of Iowa, 1997
If 0 machines are in system, then 3 are busy processing jobs;
if 1 machine is in system, then 2 are busy processing jobs, etc.

Average utilization of the machines will be

\[
\frac{3 \pi_0 + 2 \pi_1 + 1 \pi_2 + 0 \pi_3}{3} = 89.3\%
\]