# Introduction to QUEUEING : M/M/c

author

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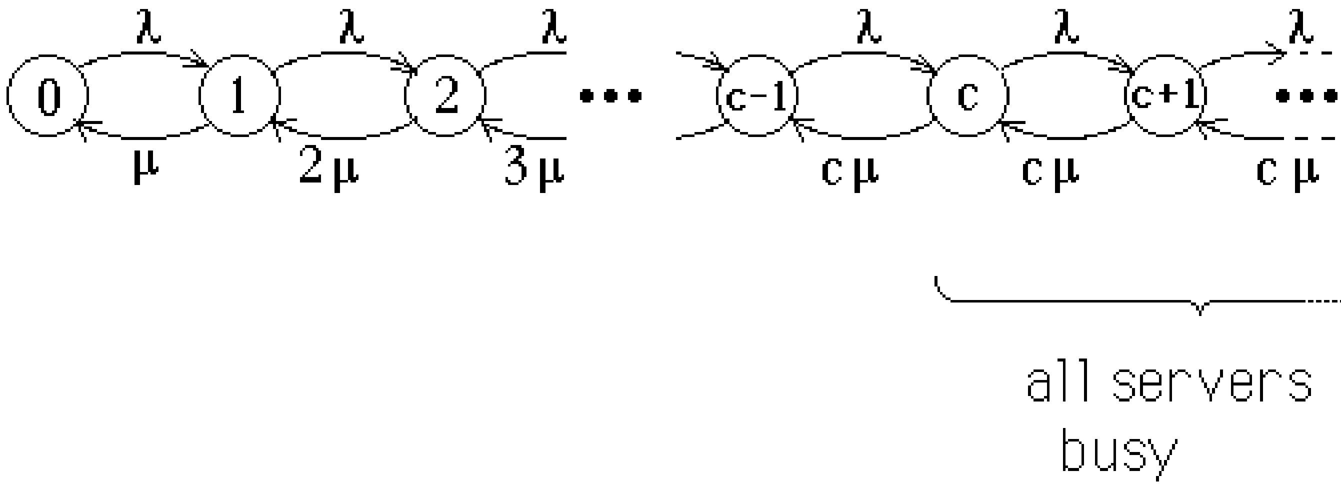
# M/M/c

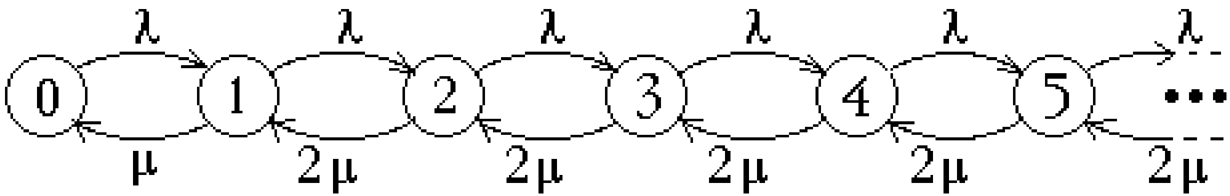
- *Arrival & Service processes are **M**emoryless, i.e.,*
  - *interarrival times have Exponential distribution with mean* $1/\lambda$
  - *service times have Exponential distribution with mean* $1/\mu$
- *Number of servers is* $c$
- *Capacity of queueing system is infinite*

# M/M/c

## Birth/Death Model



# Example: M/M/2




$$\frac{1}{\pi_0} = 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right) + \left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^3 + \cdots$$

$$= 1 + \left(\frac{\lambda}{\mu}\right)\left[1 + \left(\frac{\lambda}{2\mu}\right) + \left(\frac{\lambda}{2\mu}\right)^2 + \left(\frac{\lambda}{2\mu}\right)^3 + \cdots\right]$$

*geometric series*

$$\frac{1}{\pi_0} = 1 + \left(\frac{\lambda}{\mu}\right)\underbrace{\left[1 + \left(\frac{\lambda}{2\mu}\right) + \left(\frac{\lambda}{2\mu}\right)^2 + \left(\frac{\lambda}{2\mu}\right)^3 + \cdots\right]}$$

*geometric series*

*converges to* $\frac{1}{1 - \lambda/2\mu}$ *if* $\lambda/2\mu < 1$

$$\frac{1}{\pi_0} = 1 + \left(\frac{\lambda}{\mu}\right)\frac{1}{1 - \lambda/2\mu}$$

**M/M/c** If the arrival rate $\lambda$ is less than the combined rate $c\mu$ at which the servers can work, then the system will have a ***steadystate*** distribution, given by:

$$\pi_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!} \frac{1}{1-\rho}}$$

$$\pi_j = \frac{(c\rho)^j}{j!} \pi_0 \ , \ j=1,2,\ldots c$$

$$\pi_j = \frac{(c\rho)^j}{c!c^{j-c}} \pi_0 \ , \ j=c,c+1,\ldots.$$

where $\rho = \frac{\lambda}{c\mu} < 1$

*Probability that all servers are busy:*

$$\sum_{j \geq c}^{\infty} \pi_j = \frac{(c\rho)^c}{c!(1-\rho)} \pi_0 \quad \text{where} \quad \rho = \frac{\lambda}{c\mu} < 1$$

This, then, is the probability that an arriving customer will be required to wait for service!

# M/M/c Average Length of Queue

*(not including those being served)*

$$L_q = \sum_{j \geq c}^{\infty} (j - c)\,\pi_j \quad \text{where} \quad \pi_j = \frac{(c\rho)^j}{c!\,c^{j-c}}\,\pi_0 \;,\; j=c,c+1,\ldots.$$

$$L_q = \sum_{j=0}^{\infty} j\,\pi_{c+j} = \sum_{j=0}^{\infty} j\,\pi_0 \frac{(c\rho)^{c+j}}{c!\,c^j} = \pi_0 \frac{(c\rho)^c}{c!} \sum_{j=0}^{\infty} j\,\rho^j$$

$$\rho = \frac{\lambda}{c\,\mu}$$

$$L_q = \pi_0 \frac{(c\rho)^c}{c!} \sum_{j=0}^{\infty} j\,\rho^j = \pi_0 \frac{(c\rho)^c}{c!} \rho \underbrace{\sum_{j=0}^{\infty} j\,\rho^{j-1}}_{\text{derivative of a geometric series}}$$

$$\sum_{j=0}^{\infty} j\,\rho^{j-1} = \frac{d}{d\rho}\sum_{j=0}^{\infty} \rho^j = \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) = \frac{1}{(1-\rho)^2}$$

$$L_q = \pi_0 \frac{(c\rho)^c}{c!} \rho \frac{1}{[1-\rho]^2}$$

# Average Length of Queue

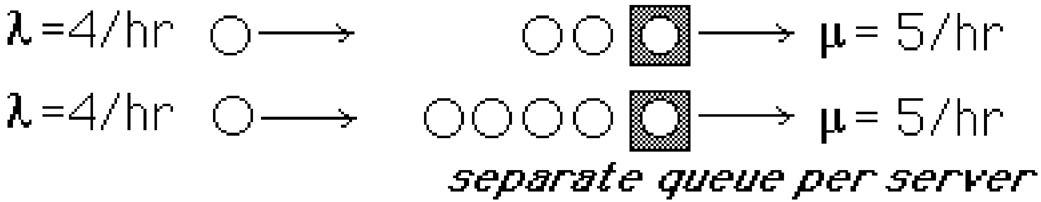
$$L_q = \frac{\rho\,(c\rho)^c}{c!}\,\pi_0 \left(\frac{1}{1-\rho}\right)^2$$

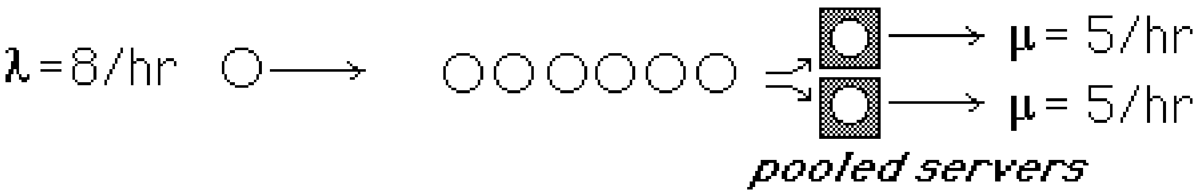
Once $L_q$ is computed, then we can compute (using Little's formula)

$$W_q = L_q/\lambda, \quad W = W_q + 1/\mu, \;\&\; L = \lambda W$$

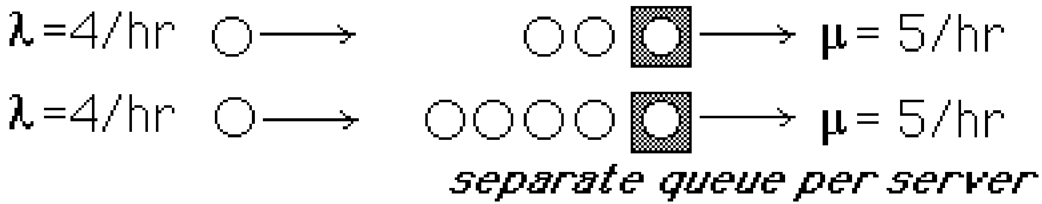
# Example: Pooled vs. Separate Servers

Compare two queueing systems:



# two M/M/1 queues

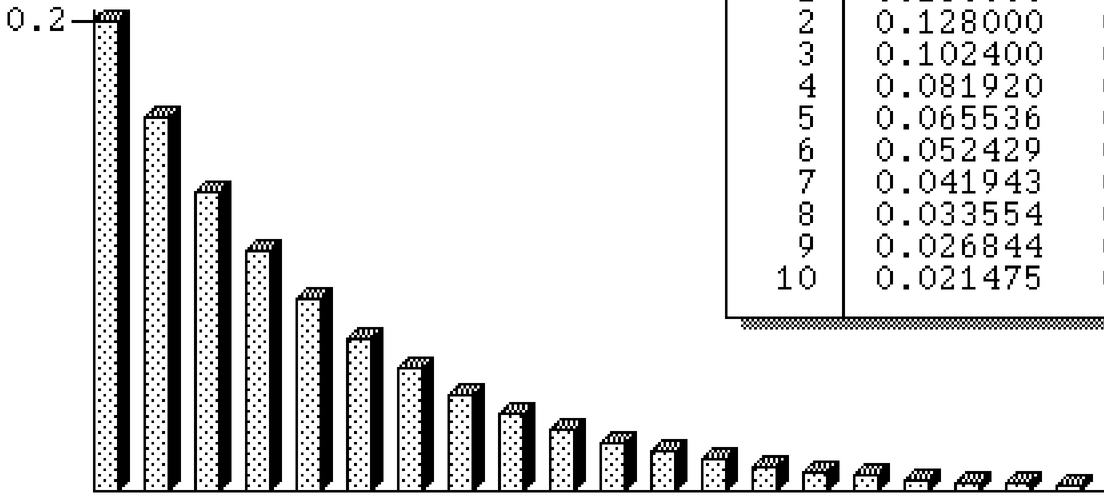



Average waiting time: $W_q = \dfrac{\lambda}{\mu\,(\mu - \lambda)}$

$$W_q = \frac{4/hr}{(5/hr)(5-4)/hr} = 0.8\ hr$$

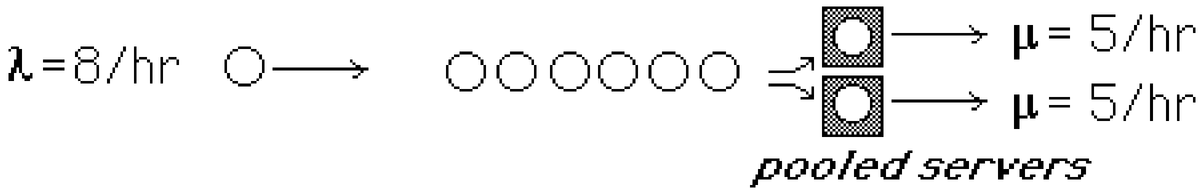
(48 minutes)

Steady-State Distribution




| i | Pi | CDF |
|---|---|---|
| 0 | 0.200000 | 0.200000 |
| 1 | 0.160000 | 0.360000 |
| 2 | 0.128000 | 0.488000 |
| 3 | 0.102400 | 0.590400 |
| 4 | 0.081920 | 0.672320 |
| 5 | 0.065536 | 0.737856 |
| 6 | 0.052429 | 0.790285 |
| 7 | 0.041943 | 0.832228 |
| 8 | 0.033554 | 0.865782 |
| 9 | 0.026844 | 0.892626 |
| 10 | 0.021475 | 0.914101 |

# single M/M/2 queue

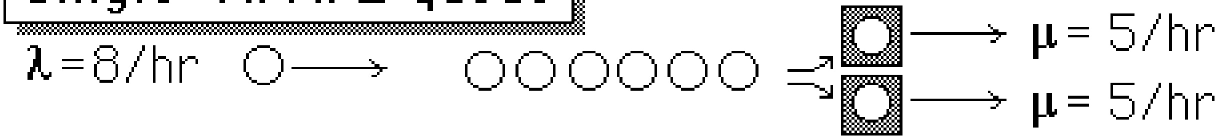



Rather than maintaining a separate queue for each server, customers enter a common queue.

$$\rho = \frac{\lambda}{2\mu} = \frac{8/\text{hr}}{2 \times 5/\text{hr}} = 0.8 < 1$$

*which implies that a steady state exists!*

## single M/M/2 queue



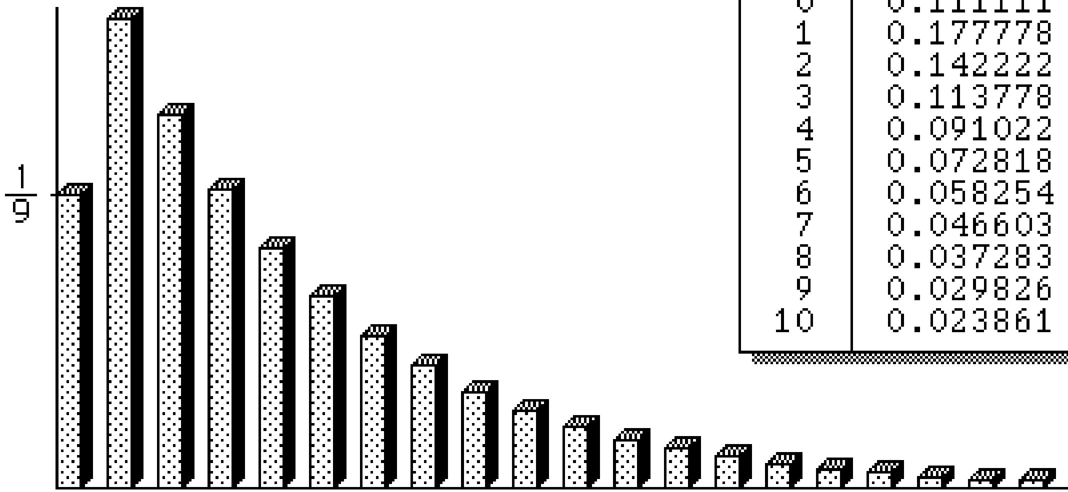

$$\pi_0 = \frac{1}{\frac{(2\times 0.8)^0}{0!} + \frac{(2\times 0.8)^1}{1!} + \frac{(2\times 0.8)^2}{2!\,(1-0.8)}} = \frac{1}{1+1.6+6.4} = \frac{1}{9}$$

$$\pi_0 = 0.111111$$

$$\pi_1 = \frac{(c\rho)^1}{1!}\pi_0 = \frac{(2\times 0.8)^1}{1!}\frac{1}{9} = 0.1777777$$

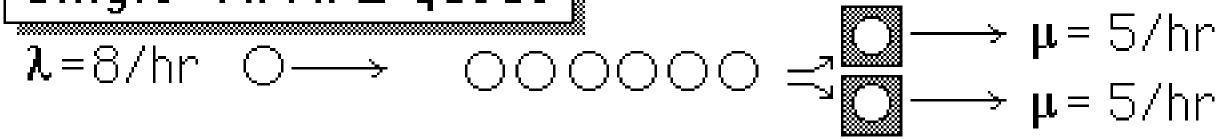
$$P\{\text{both servers busy}\} = 1 - \pi_0 - \pi_1 = 0.7111111$$

# Steady-State Distribution




| i | Pi | CDF |
|---|---|---|
| 0 | 0.111111 | 0.111111 |
| 1 | 0.177778 | 0.288889 |
| 2 | 0.142222 | 0.431111 |
| 3 | 0.113778 | 0.544889 |
| 4 | 0.091022 | 0.635911 |
| 5 | 0.072818 | 0.708729 |
| 6 | 0.058254 | 0.766983 |
| 7 | 0.046603 | 0.813586 |
| 8 | 0.037283 | 0.850869 |
| 9 | 0.029826 | 0.880695 |
| 10 | 0.023861 | 0.904556 |

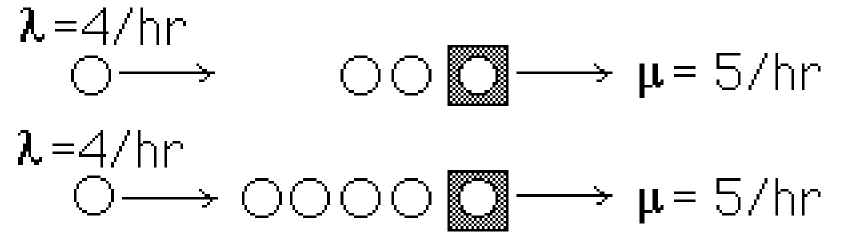
# single M/M/2 queue




$$L_q = \frac{\rho}{1-\rho} P\{\text{both servers busy}\}$$

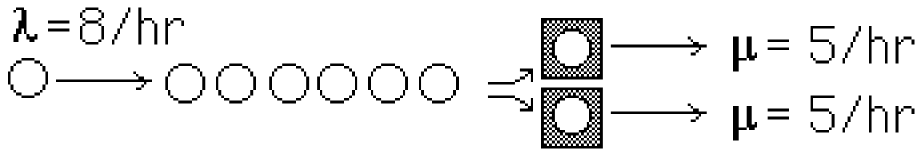
$$= \frac{0.8}{0.2} (0.7111111) = 2.844444444$$

$$W_q = \frac{L_q}{\lambda} = 0.35156 \text{ hr.} = 21.1 \text{ minutes}$$




*separate queue per server*

$W_q$ = 0.8 hr.
= 48 min.




*pooled servers*

$W_q$ = 0.352 hr.
= 21.1 min.

*By pooling the servers, the average waiting time per customer is reduced by approximately 56%*