Queueing Networks-- An Introduction

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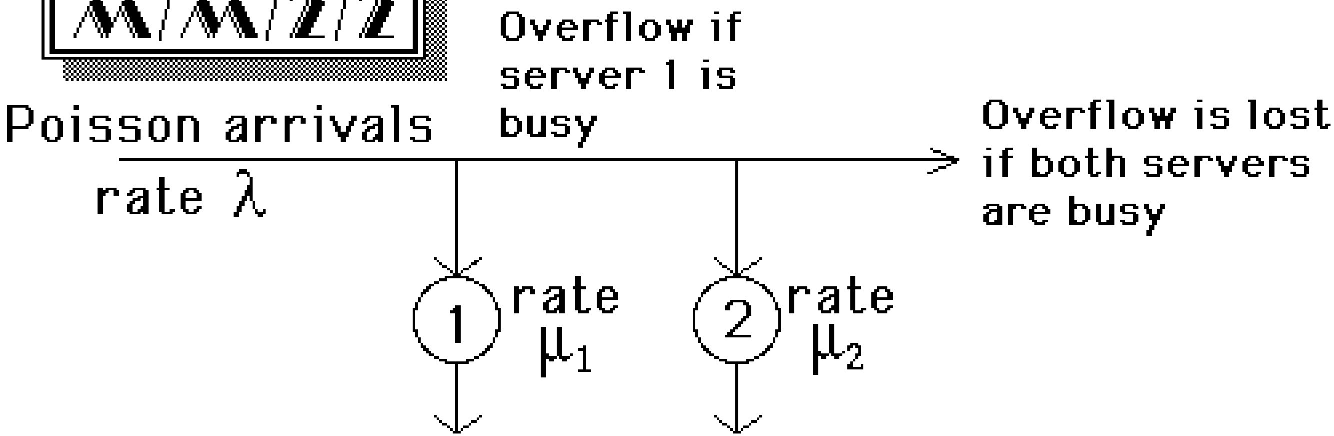
author

- M/M/2/2 Queueing System
- Two Tandem Servers w/o Queues
- Two Tandem Servers w/intervening Queue
- M/M/1 System with feedback
- Two Cyclic Queues
- Two Tandem Queues w/infinite capacity

M/M/2/2



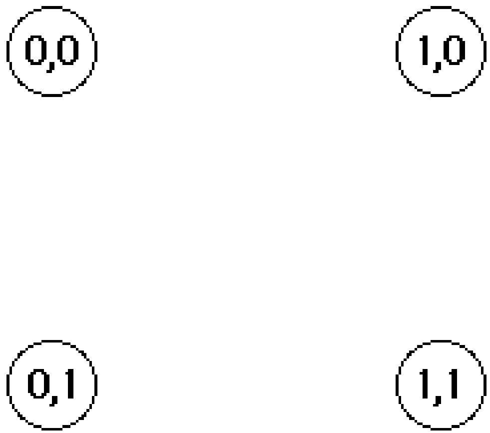
We want to compute:

- *steady-state distribution*
- *fraction of customers lost*
- *utilization of each server*

States

(n_1, n_2) where

$$n_i = \begin{cases} 1 \text{ if server } i \text{ is busy} \\ 0 \text{ otherwise} \end{cases}$$

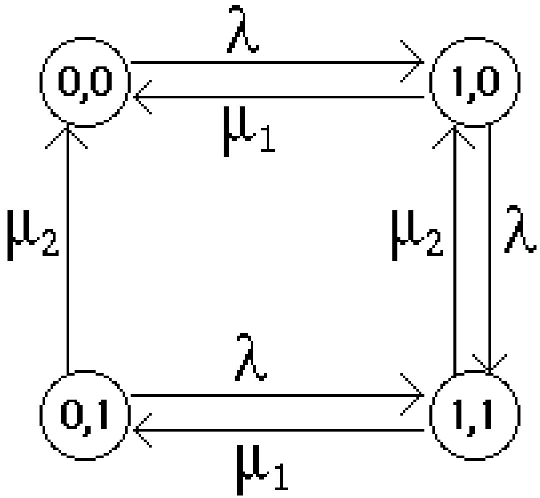


What are the transition rates?

States

(n_1, n_2) where

$$n_i = \begin{cases} 1 \text{ if server } i \text{ is busy} \\ 0 \text{ otherwise} \end{cases}$$



Transition rate matrix

from \ to	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	¯0.2	0.2	0	0
(1,0)	0.33333	¯0.53333	0	0.2
(0,1)	0.25	0	¯0.45	0.2
(1,1)	0	0.25	0.3333	¯0.5833

parameters:

$$\lambda = 1/5$$

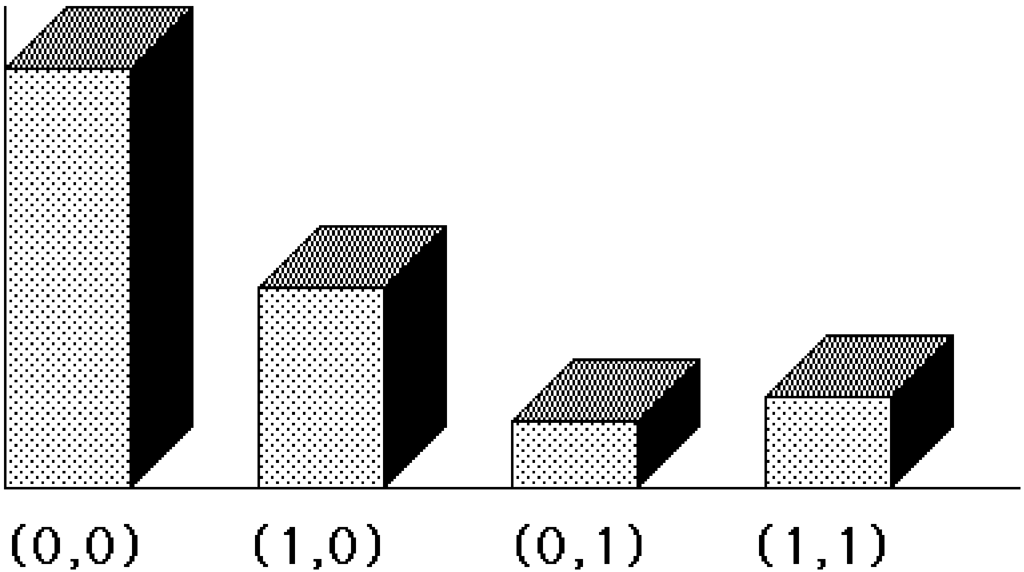
$$\mu_1 = 1/3$$

$$\mu_2 = 1/4$$

Steadystate Distribution

i	state	Pi
1	(0,0)	0.537536
2	(1,0)	0.256924
3	(0,1)	0.0874636
4	(1,1)	0.118076

Steady-State Distribution



```
PI
0.191205 0.449331 0.152964 0.206501
```

Average # in System

$$L = \sum_{i \in S} n_i \pi_i$$

```
PI+.×0 1 1 2
0.580539
```

Average Arrival Rate

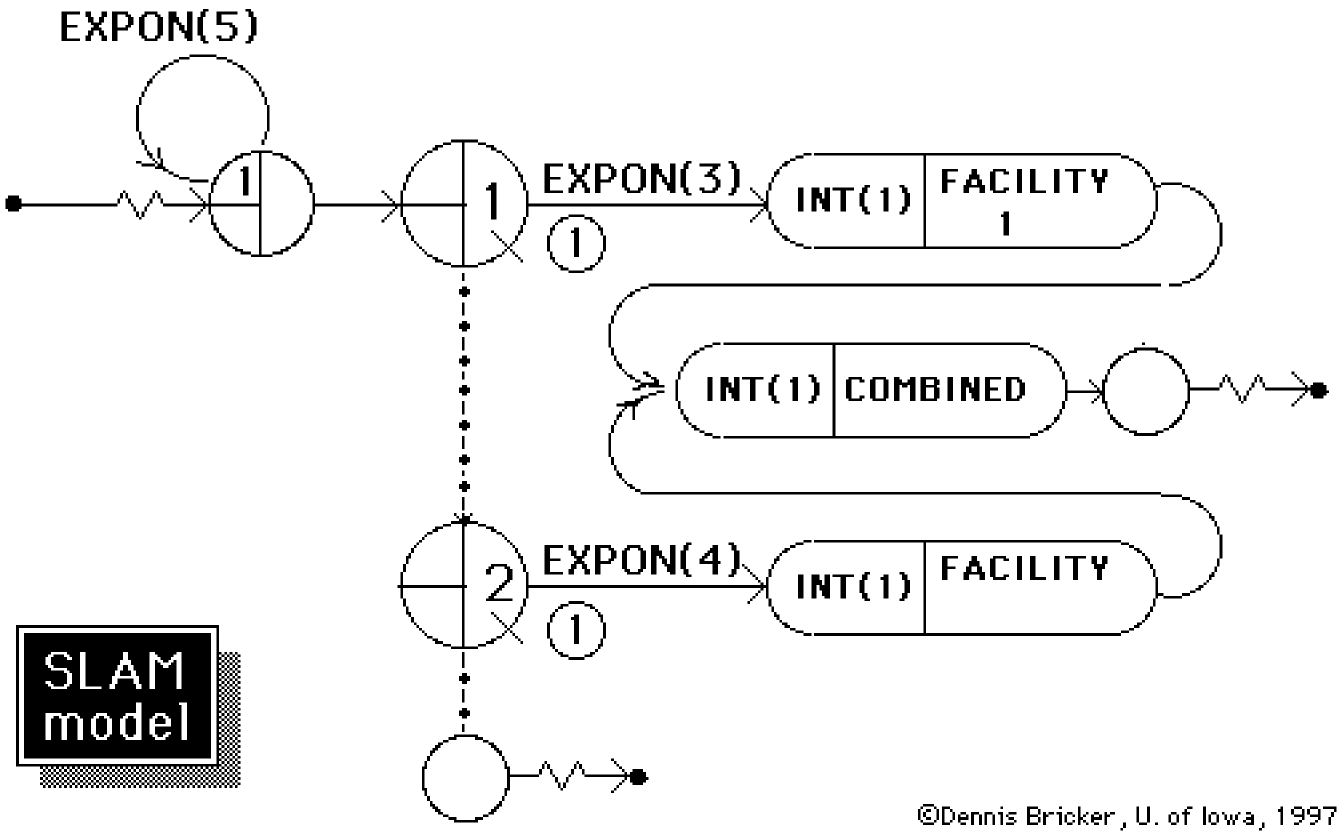
$$\bar{\lambda} = \sum_{i \in S} \lambda_i \pi_i$$

```
PI+.× .2 .2 .2 0
0.176385
```

Average Time in System

$$W = L/\bar{\lambda}$$

```
(PI+.×0 1 1 2)÷ PI+.× .2 .2 .2 0
3.29132
```


EXPON(5)
1
1
EXPON(3)
1
INT(1)
FACILITY
1
INT(1)
COMBINED
2
EXPON(4)
1
INT(1)
FACILITY
SLAM
model

```
GEN,BRICKER,M_M_2_2,4/14/93,1,Y,Y,Y/N,Y,Y,72;
LIM,2,2,100;
INIT,,3700;
MONTR,CLEAR,100
NETWORK;
      CREATE,EXPON(5.0),,1;
Q1    QUEUE(1),,0,BALK(Q2);
      ACTIVITY/1,EXPON(3.0); FACILITY 1
      COLCT(1),INT(1),TIME_IN_SYS_1;
      ACT,,,C3;
Q2    QUEUE(2),,0,BALK(T);
      ACTIVITY/2,EXPON(4.0); FACILITY 2
      COLCT(2),INT(1),TIME_IN_SYS_2;
C3    COLCT(3),INT(1),TIME_IN_SYS_3;
T     TERM;
      END;
FIN;
```

SLAM code

STATISTICS FOR VARIABLES BASED ON OBSERVATION

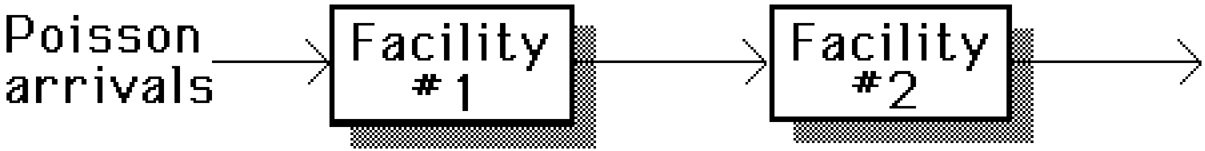
	MEAN VALUE	STANDARD DEVIATION	COEFF. OF VARIATION	MINIMUM VALUE	MAXIMUM VALUE	NO.OF OBS
TIME_IN_SYS_1	0.288E+01	0.291E+01	0.101E+01	0.110E-02	0.352E+02	459
TIME_IN_SYS_2	0.367E+01	0.386E+01	0.105E+01	0.234E-01	0.253E+02	186
TIME_IN_SYS_3	0.311E+01	0.323E+01	0.104E+01	0.110E-02	0.352E+02	645

SERVICE ACTIVITY STATISTICS

ACT NUM	ACT LABEL OR START NODE	SER CAP	AVERAGE UTIL	STD DEV	CUR UTIL	AVERAGE BLOCK	MAX IDL TME/SER	MAX BSY TME/SER	ENT CNT
1	FACILITY 1	1	0.368	0.48	0	0.00	27.63	35.22	459
2	FACILITY 2	1	0.187	0.39	0	0.00	94.70	25.27	186

Comparison	Simulation	Markov Chain Analysis
Average # in System	0.555	0.581
Average Time in System	3.11	3.29

2 Tandem Servers, with no queues



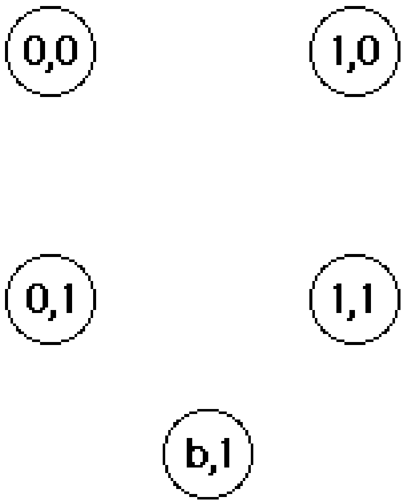
- 2 identical servers, with exponentially dist'd service times
- No queues allowed in front of either server
- Server 1 is "blocked" whenever it has completed service while server 2 is busy
- Arrivals at #1 are turned away when it is busy or blocked

States

(x,y) where

$$x = \begin{cases} 0 \text{ if } \#1 \text{ is idle} \\ 1 \text{ if } \#1 \text{ is busy} \\ b \text{ if } \#1 \text{ is blocked} \end{cases}$$

$$y = \begin{cases} 0 \text{ if } \#2 \text{ is idle} \\ 1 \text{ if } \#2 \text{ is busy} \end{cases}$$

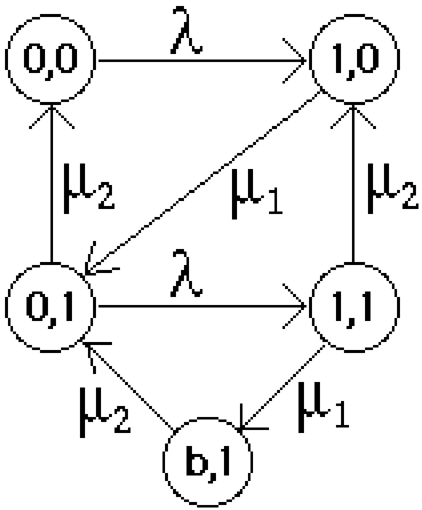


States

(x,y) where

$$x = \begin{cases} 0 \text{ if \#1 is idle} \\ 1 \text{ if \#1 is busy} \\ b \text{ if \#1 is blocked} \end{cases}$$

$$y = \begin{cases} 0 \text{ if \#2 is idle} \\ 1 \text{ if \#2 is busy} \end{cases}$$



Tandem Servers w/o queues

Transition rate matrix

parameters:

$\lambda = 4$

$\mu_1 = 4$

$\mu_2 = 5$

from \ to	1	2	3	4	5
1	−4	4	0	0	0
2	0	−4	4	0	0
3	5	0	−9	4	0
4	0	5	0	−9	4
5	0	0	5	0	−5

Tandem Servers w/o queues

Steadystate Distribution

i	state	Pi
1	(0,0)	0.257437
2	(1,0)	0.371854
3	(0,1)	0.20595
4	(1,1)	0.0915332
5	(b,1)	0.0732265

Average # in System

$$L = \sum_{i \in S} n_i \pi_i$$

```
      PI+.× 0 1 1 2 2
0.907323
```

Average Arrival Rate

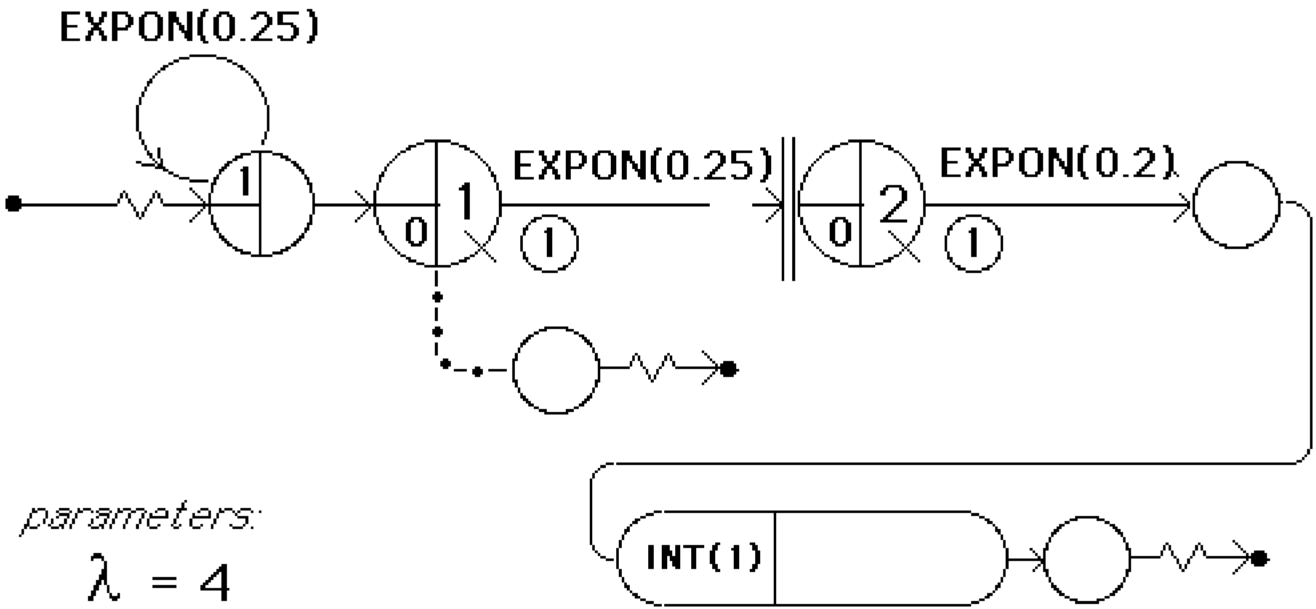
$$\bar{\lambda} = \sum_{i \in S} \lambda_i \pi_i$$

```
      PI+.× 4 0 4 0 0
1.85355
```

Average Time in System

$$W = L / \bar{\lambda}$$

```
      (PI+.×0 1 1 2 2)÷PI+.× 4 0 4 0 0
0.489506
```



parameters:

$\lambda = 4$

$\mu_1 = 4$

$\mu_2 = 5$

SLAM model

```
GEN,BRICKER,TANDEM,4/14/93,1,Y,Y,Y/N,Y,Y,72;
LIM,2,2,100;
INIT,,3700;
MONTR,CLEAR,100
NETWORK;
      CREATE,EXPON(0.25),,1;
Q1    QUEUE(1),,0,BALK(T);
      ACTIVITY/1,EXPON(0.25); FACILITY 1
Q2    QUEUE(2),,0,BLOCK;
      ACTIVITY/2,EXPON(0.20); FACILITY 2
      COLCT(1),INT(1),TIME_IN_SYS;
T     TERM;
      END;
FIN;
```

SLAM code

STATISTICS FOR VARIABLES BASED ON OBSERVATION

	MEAN VALUE	STANDARD DEVIATION	COEFF. OF VARIATION	MINIMUM VALUE	MAXIMUM VALUE	NO.OF OBS
TIME_IN_SYS	0.481E+00	0.322E+00	0.671E+00	0.586E-02	0.287E+01	6708

FILE STATISTICS

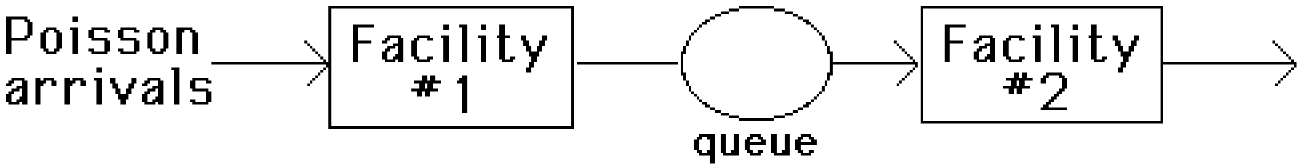
FILE NUMBER	LABEL/TYPE		AVERAGE LENGTH	STANDARD DEVIATION	MAXIMUM LENGTH	CURRENT LENGTH	AVERAGE WAIT TIME
1	Q1	QUEUE	0.000	0.000	0	0	0.000
2	Q2	QUEUE	0.000	0.000	0	0	0.000

SERVICE ACTIVITY STATISTICS

ACT NUM	ACT LABEL OR START NODE	SER CAP	AVERAGE UTIL	STD DEV	CUR UTIL	AVERAGE BLOCK	MAX IDL TME/SER	MAX BSY TME/SER	ENT CNT
1	FACILITY 1	1	0.463	0.50	1	0.07	2.43	2.42	6708
2	FACILITY 2	1	0.364	0.48	0	0.00	2.78	1.71	6708

Comparison	Simulation	Markov Chain Analysis
Average # in System	0.827	0.907
Average Time in System	0.481	0.489

Suppose that we add space for 1 customer to wait between the two facilities...

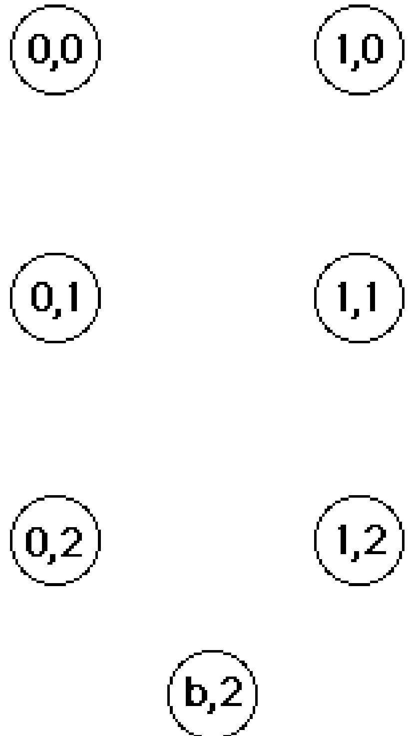


States

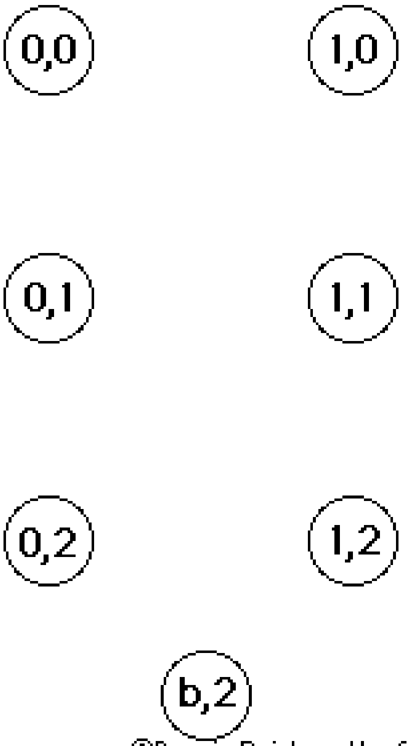
(x,y) where

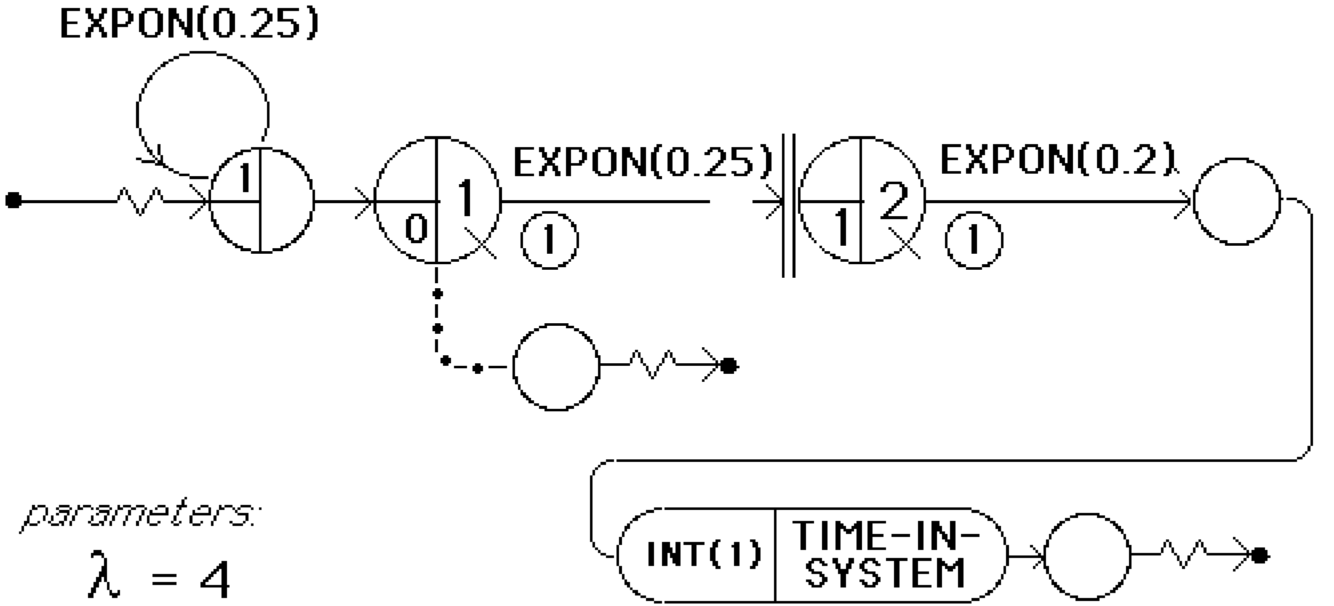
x = { 0 if #1 is idle; 1 if #1 is busy; b if #1 is blocked }

y = { 0 if #2 is idle; 1 if #2 is busy; 2 if customer waits between }



Transition Diagram





parameters:

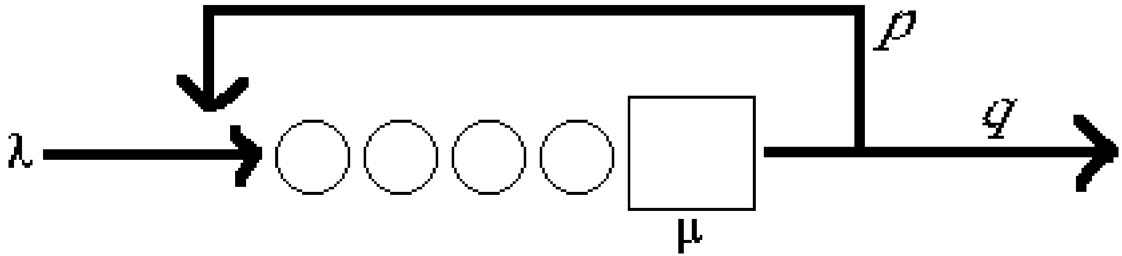
$\lambda = 4$

$\mu_1 = 4$

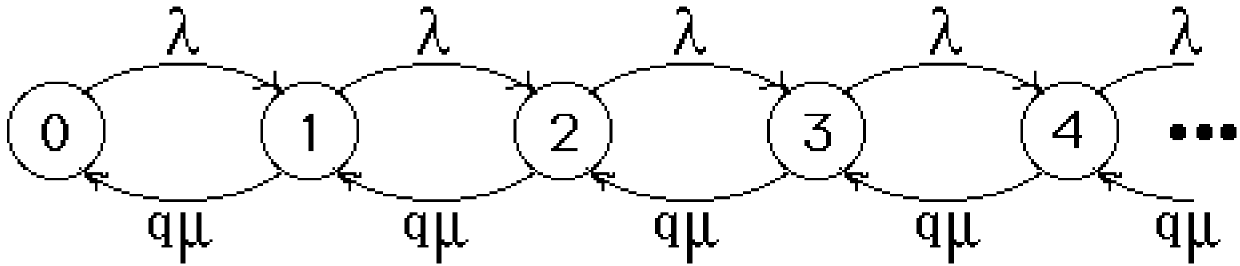
$\mu_2 = 5$

SLAM model

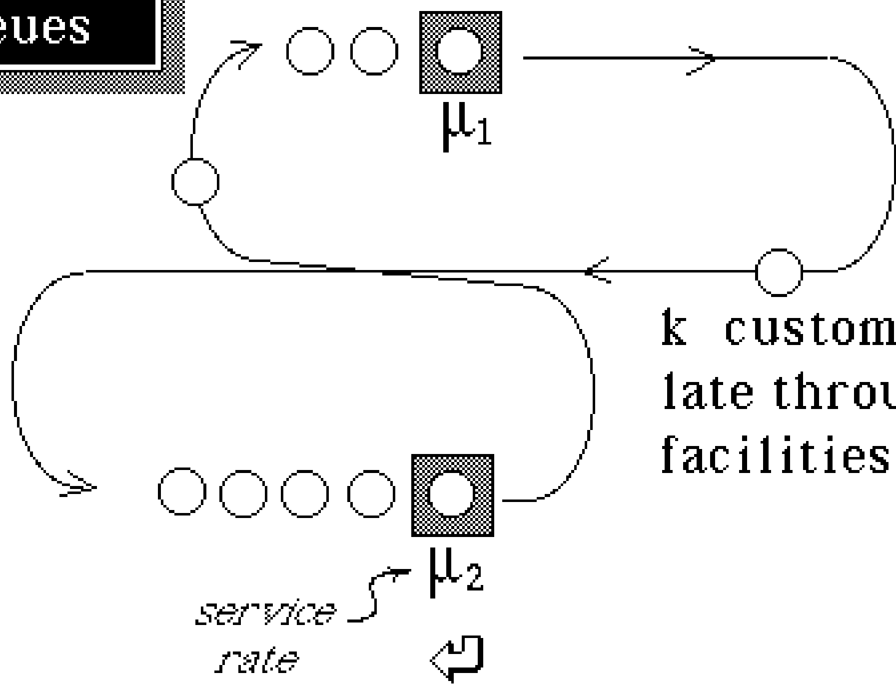
M/M/1 System with feedback



A customer, when service is complete, will depart with probability q and return to end of queue with probability $p=1-q$


λ
λ
λ
λ
λ
0
1
2
3
4
•••
qμ
qμ
qμ
qμ
qμ

Two Cyclic Queues



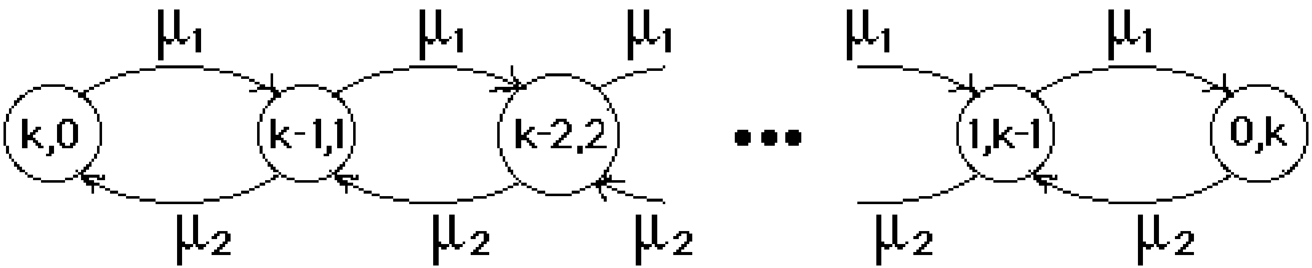
k customers circulate through two facilities

States

(i,j) where

i = # customers in subsystem 1

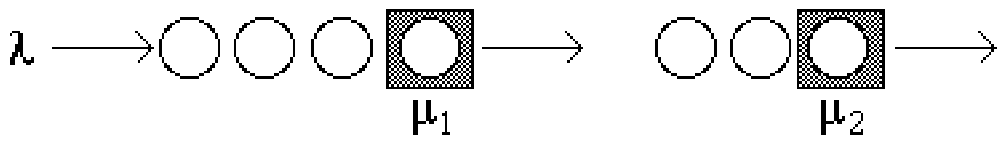
j = # customers in subsystem 2



Is this a birth-death process?

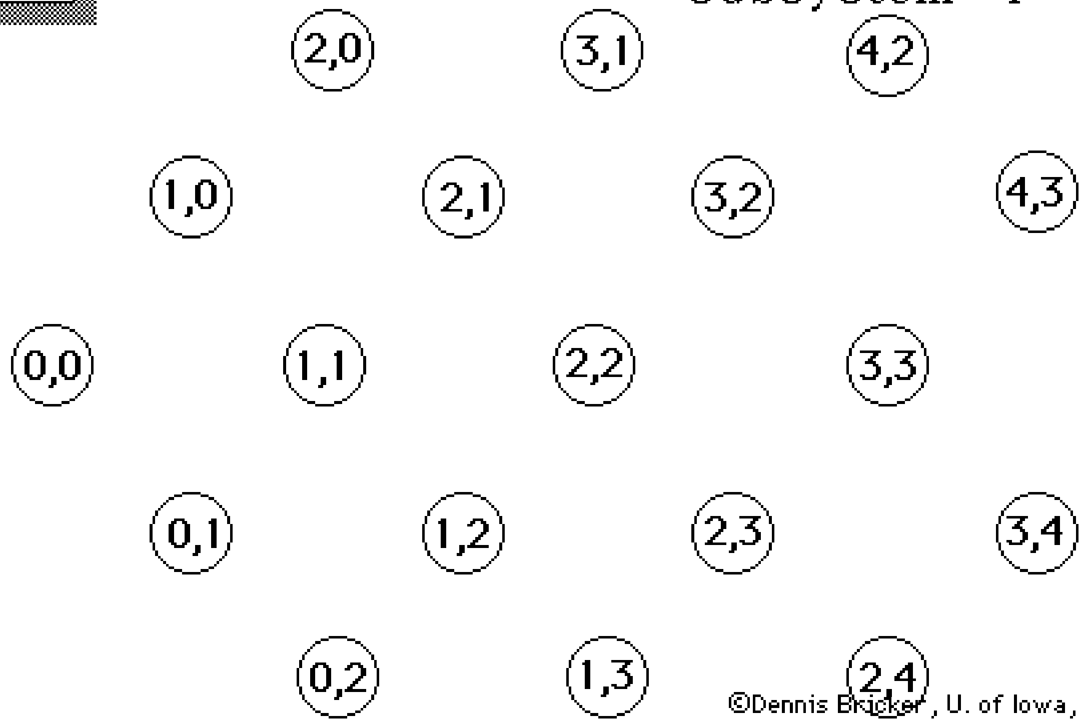
Compare with M/M/1/k queueing system!

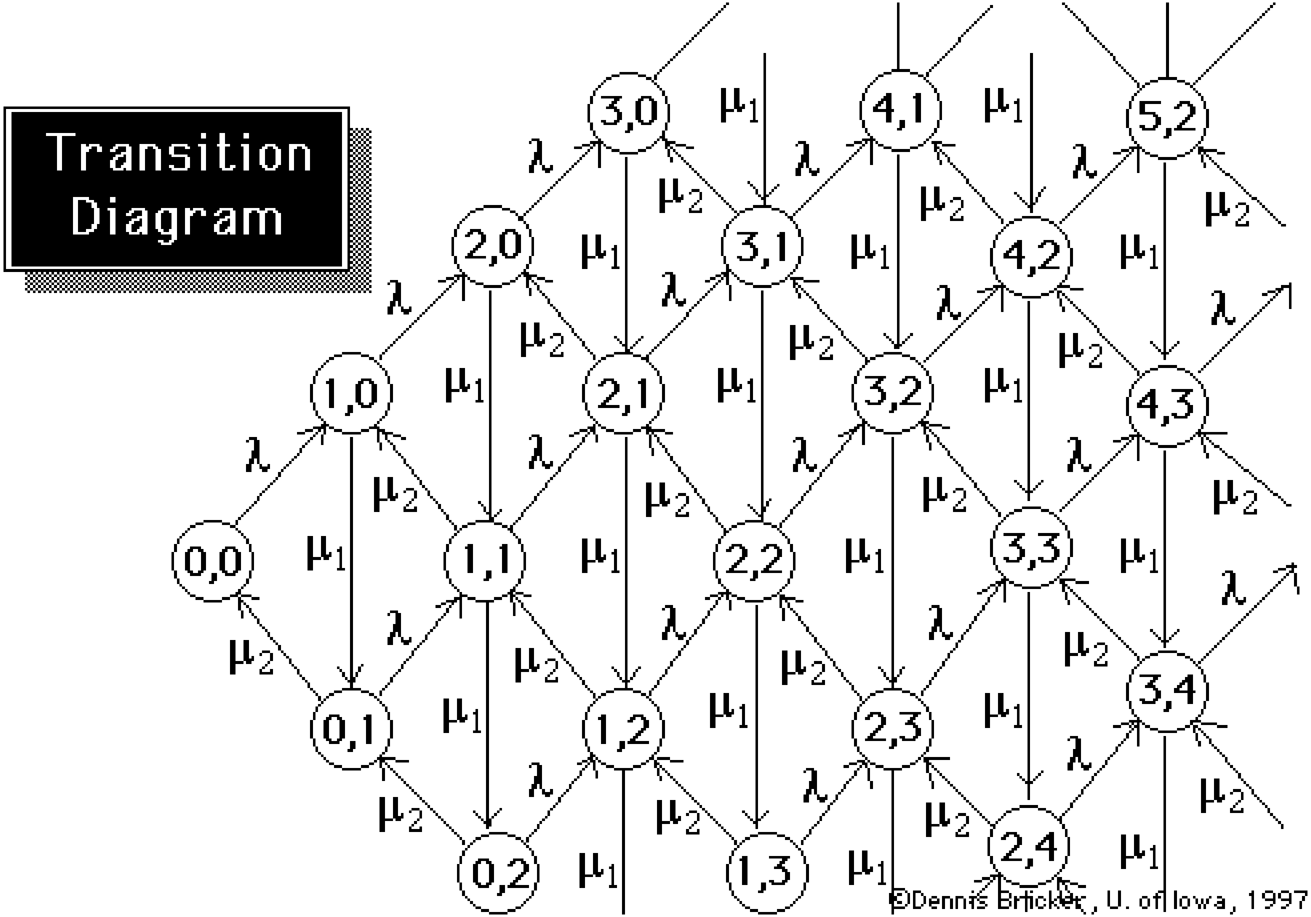
Two Tandem Queues with infinite capacity



States

(m_1, m_2) where m_i = #customers in subsystem #i




Transition Diagram
0,0
1,0
0,1
2,0
1,1
0,2
3,0
2,1
1,2
3,1
2,2
1,3
4,1
3,2
2,3
4,2
3,3
2,4
5,2
4,3
3,4
λ
μ_1
μ_2
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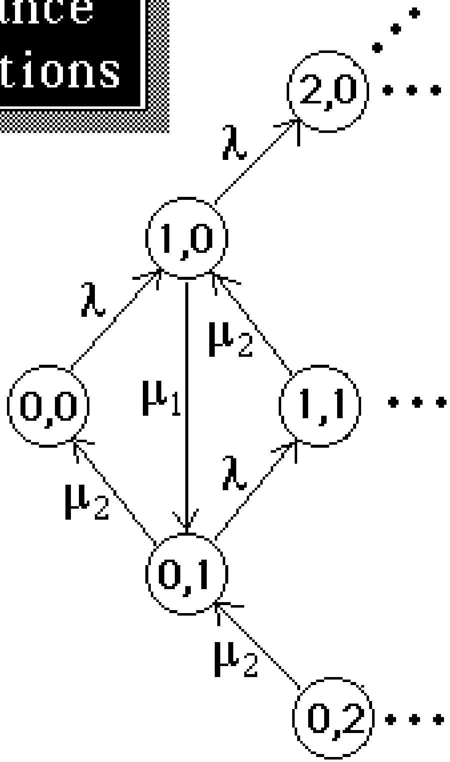
Balance Equations



for state 0,0

$$\lambda \pi_{00} = \mu_2 \pi_{01}$$

for state 1,0

$$(\lambda + \mu_1)\, \pi_{10} = \lambda\, \pi_{00} + \mu_2 \pi_{11}$$

for state 0,1

$$(\lambda + \mu_2)\, \pi_{01} = \mu_1\, \pi_{10} + \mu_2 \pi_{02}$$

etc.

Balance Equations

$$\lambda \pi_{00} = \mu_2 \pi_{01}$$
$$(\lambda+\mu_1)\, \pi_{10} = \lambda\, \pi_{00} + \mu_2 \pi_{11}$$
$$(\lambda+\mu_2)\, \pi_{01} = \mu_1\, \pi_{10} + \mu_2 \pi_{02}$$
$$(\lambda+\mu_1)\, \pi_{20} = \lambda\, \pi_{10} + \mu_2 \pi_{21}$$
$$(\lambda+\mu_1+\mu_2)\, \pi_{11} = \lambda\, \pi_{01} + \mu_1\, \pi_{20} + \mu_2 \pi_{12}$$
$$(\lambda+\mu_2)\, \pi_{02} = \mu_1\, \pi_{11} + \mu_2 \pi_{03}$$
$$\vdots$$

We get infinitely many equations in infinitely many unknowns!

Claim: these balance equations are satisfied by

$$\pi_{m_1,m_2} = \pi^1_{m_1} \times \pi^2_{m_2}$$

where

$$\pi^i_{m_i} = (1-\rho_i)\,\rho_i^{m_i}, \qquad \rho_i = \lambda/\mu_i$$

is the steady-state distribution of the M/M/1 queue

That is,

P{m_1 at station 1 & m_2 at station 2}
= P{m_1 at station 1} × P{m_2 at station 2}

$$\pi_{m_1,m_2} = \pi^1_{m_1} \times \pi^2_{m_2}$$

$$\pi^i_{m_i} = (1-\rho_i)\,\rho_i^{m_i}\ ,\ \rho_i = \lambda/\mu_i$$

Substituting into

$$\lambda\pi_{00} = \mu_2\pi_{01}$$

yields $\lambda\left(1-\lambda/\mu_1\right)\left(1-\lambda/\mu_2\right) = \mu_2\left(1-\lambda/\mu_1\right)\left(1-\lambda/\mu_2\right)\lambda/\mu_2$

Substituting into $(\lambda+\mu_1)\,\pi_{10} = \lambda\,\pi_{00} + \mu_2\pi_{11}$

yields

$$(\lambda+\mu_1)\left(1-\lambda/\mu_1\right)\lambda/\mu_1\left(1-\lambda/\mu_2\right) =$$
$$\lambda\left(1-\lambda/\mu_1\right)\left(1-\lambda/\mu_2\right) + \mu_2\left(1-\lambda/\mu_1\right)\lambda/\mu_1\left(1-\lambda/\mu_2\right)\lambda/\mu_2$$

etc.