the Poisson process

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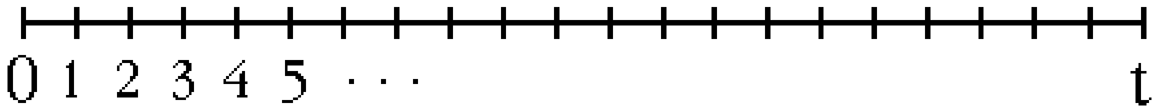
The Poisson Process as a limiting case of the Bernoulli Process

Consider the following situation:



A time interval of length t seconds is divided into one-second intervals, with the probability of a vehicle arriving at an intersection during a one-second interval being a small number p. (Assume that the probability that more than one vehicle arrives is negligible.)

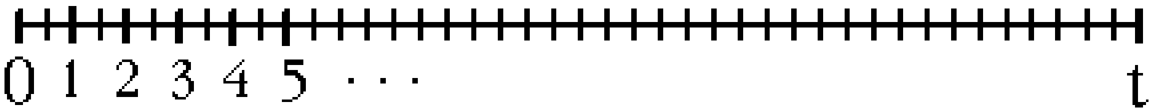
Consider the Bernoulli process $\{X_k\,;\ k=1,2,\ldots\}$ where $X_k = 1$ if a vehicle arrives during the k^{th} second, and the associated counting process $\{N_t\}$ which counts the number of arrivals during the interval $[0,t]$.

Then N_t has the binomial distribution:

$$P\{N_t = x\} = \binom{t}{x} p^x (1-p)^{n-x}$$

with expected value $\nu = tp$.

Consider what happens as we divide [0,t] into n smaller time intervals, but in such as way that the expected number of arrivals in [0,t] remains constant, ν.



That is, the probability of an arrival in each of these small intervals must be ν/n , and

$$P\{N_t = x\} = \binom{n}{x}\left(\frac{\nu}{n}\right)^x\left(1-\frac{\nu}{n}\right)^{n-x}$$

$$P\{N_t = x\} = \binom{n}{x}\left(\frac{\nu}{n}\right)^x\left(1 - \frac{\nu}{n}\right)^{n-x}$$

$$= \left(\frac{n!}{x!(n-x)!}\right)\left(\frac{\nu^x}{n^x}\right)\left(1 - \frac{\nu}{n}\right)^n\left(1 - \frac{\nu}{n}\right)^{-x}$$

$$= \frac{\nu^x}{x!}\left(1 - \frac{\nu}{n}\right)^n\left(\frac{n!}{(n-x)!}\right)\frac{1}{n^x\left(1 - {}^{\nu}\!/_{n}\right)^x}$$

Consider the limit of this distribution as $n \longrightarrow +\infty$

$$P\{N_t = x\} = \frac{\nu^x}{x!}\underbrace{\left(1 - \frac{\nu}{n}\right)^n}_{\downarrow \; e^{-\nu}} \underbrace{\left(\frac{n!}{(n-x)!}\right)\frac{1}{n^x\left(1 - \nu/n\right)^x}}$$

$$\left(\frac{n!}{(n-x)!}\right)\frac{1}{n^x\left(1 - \nu/n\right)^x} = \underbrace{\frac{n(n-1)(n-2)\ \dots\ (n-x+1)}{\left[n\left(1-\nu/n\right)\right]^x}}_{\longrightarrow \frac{n^x}{n^x} = 1}$$

$$P\{N_t = x\} = \frac{\nu^x}{x!} e^{-\nu}$$

If the arrival rate is λ /second, then $\nu = \lambda t$ and

$$P\{N_t = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

for x=0, 1, 2, 3, ...

Poisson Distribution

Poisson Distribution

$$P\{N_t = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

for $x=0, 1, 2, 3, \ldots$

Mean Value

$$E(N_t) = \lambda t$$

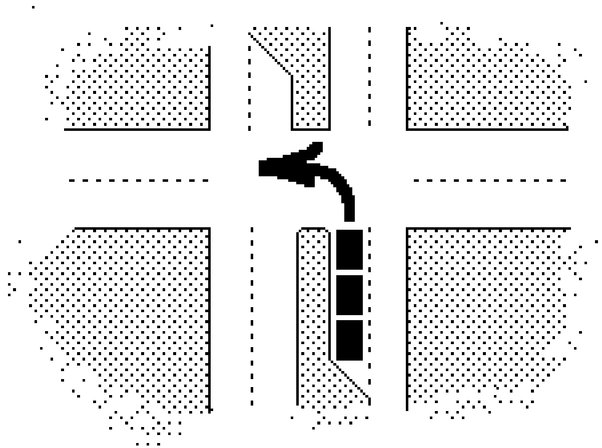
Variance

$$Var(N_t) = \lambda t$$

mean and variance are equal!

Example A left-turn lane at an intersection has a capacity of 3 autos. 30% of autos arriving at the intersection wish to turn left.
The ***expected number*** of autos arriving during a red signal is 6.

What is the probability that the capacity of the left-turn lane is exceeded during a red signal?



Given that **N** autos arrive, the number **X** of left-turning autos has the ***binomial*** distribution.

The number **N** of autos arriving during the red signal has the ***Poisson*** distribution.

$$P\{X>3\} = \sum_{N=4}^{\infty} \underbrace{P\{X>3 \mid N \text{ arrivals}\}}_{\text{computed using binomial distn.}} \; \underbrace{P\{N \text{ arrivals}\}}_{\text{computed using Poisson distn.}}$$

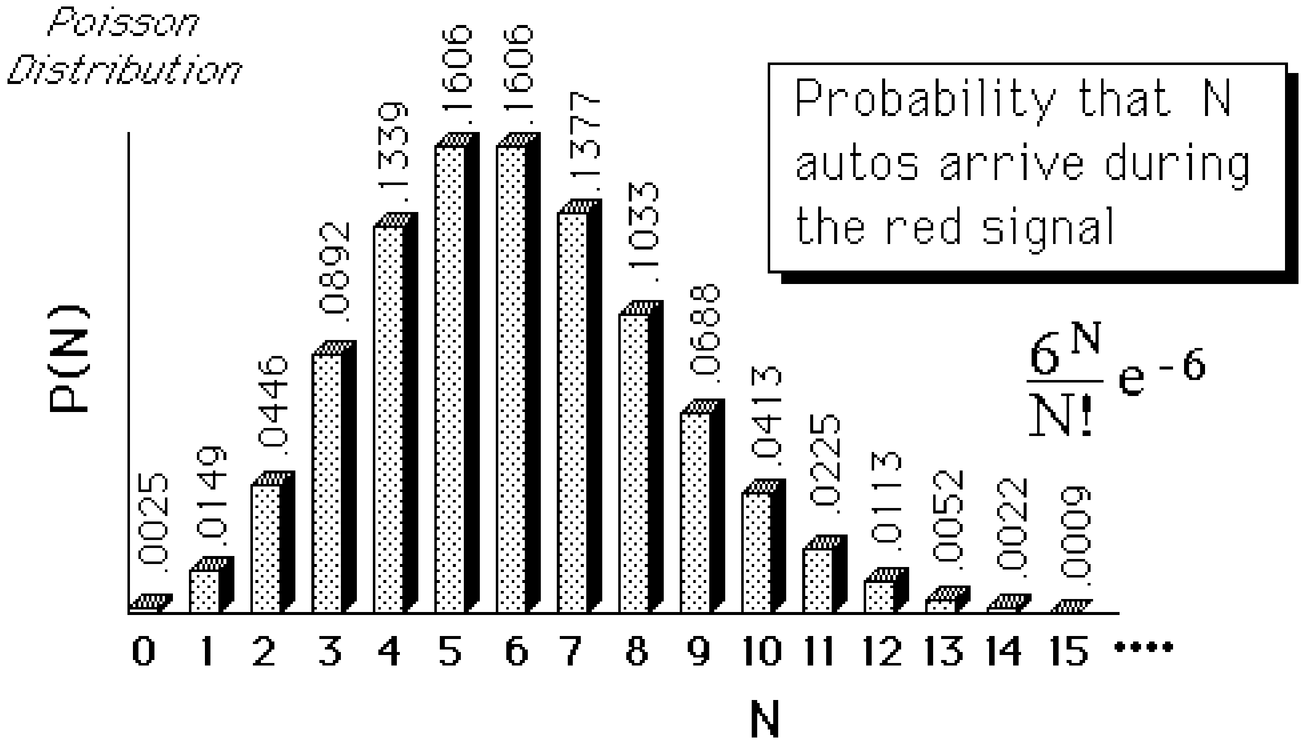
$$P\{X>3\,|\,N\text{ arrivals}\} = \sum_{x=4}^{N}\binom{N}{x}(0.3)^x(0.7)^{N-x}$$

binomial distn.

$$= 1 - \sum_{x=0}^{3}\binom{N}{x}(0.3)^x(0.7)^{N-x}$$

$$P\{N\text{ arrivals}\} = \frac{6^N}{N!}e^{-6}$$

Poisson distn.



Poisson Distribution
P(N)
.0025
.0149
.0446
.0892
.1339
.1606
.1606
.1377
.1033
.0688
.0413
.0225
.0113
.0052
.0022
.0009
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ••••
N
Probability that N autos arrive during the red signal
$\frac{6^N}{N!}e^{-6}$

N	P{N}	P{X|N}	P{X|N}P{N}
0	0.00247875	0.00000000	0.00000000
1	0.01487251	0.00000000	0.00000000
2	0.04461754	0.00000000	0.00000000
3	0.08923508	0.00000000	0.00000000
4	0.13385262	0.00810000	0.00108421
5	0.16062314	0.03078000	0.00494398
6	0.16062314	0.07047000	0.01131911
7	0.13767698	0.12603600	0.01735226
8	0.10325773	0.19410435	0.02004278
9	0.06883849	0.27034090	0.01860986
10	0.04130309	0.35038928	0.01447216
11	0.02252896	0.43043766	0.00969731
12	0.01126448	0.50748423	0.00571655
13	0.00519899	0.57939435	0.00301227
14	0.00222814	0.64483257	0.00143678
15	0.00089126	0.70313207	0.00062667
⋮	⋮	⋮	⋮
			0.1083

The probability that the capacity of the left-turn lane is exceeded during each red signal is about 11%

Time between arrivals

Suppose that the number of arrivals in an interval has the Poisson distribution with arrival rate λ/second.

Let T_1 = time of the first arrival.

What is the distribution of T_1 ?

$$P\{T_1 > t\} = P\{\text{ NO arrivals occur in interval } [0,t]\}$$

$$= \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$$

← Poisson distribution $P\{N_t = 0\}$

CDF: $P\{T_1 \le t\} = F(t) = 1 - e^{-\lambda t}$

Density function: $f(t) = \frac{d}{dt} F(t) = \lambda e^{-\lambda t}$

Exponential Distribution

Exponential Distribution

$$F(t) = 1 - e^{-\lambda t}$$

Mean Value

$$E(T_1) = \frac{1}{\lambda}$$

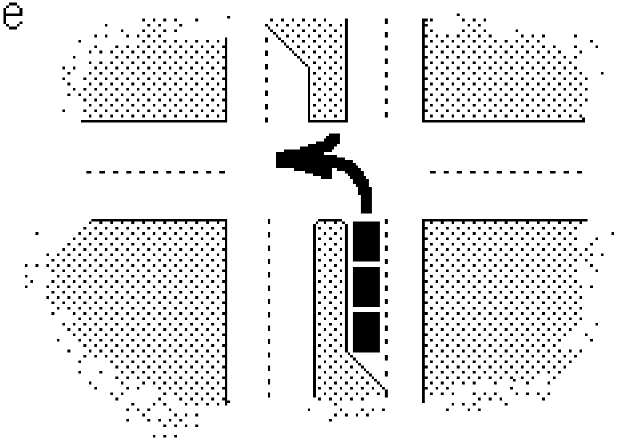
Variance

$$Var(T_1) = \frac{1}{\lambda^2}$$

Example

Suppose that the arrival rate for northbound autos is 6 per 30 second red signal, i.e., 0.2/second

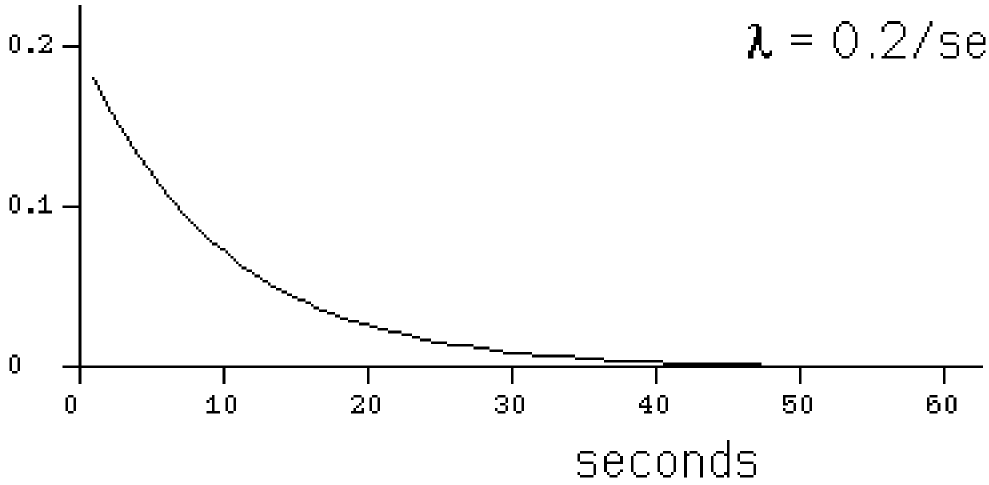
What is the distribution of the arrival time of the first auto?
(This will also be the distribution of the time between arrivals!)



Exponential Distribution

$$f(t) = \lambda e^{-\lambda t}$$

$\lambda = 0.2/\text{sec.}$

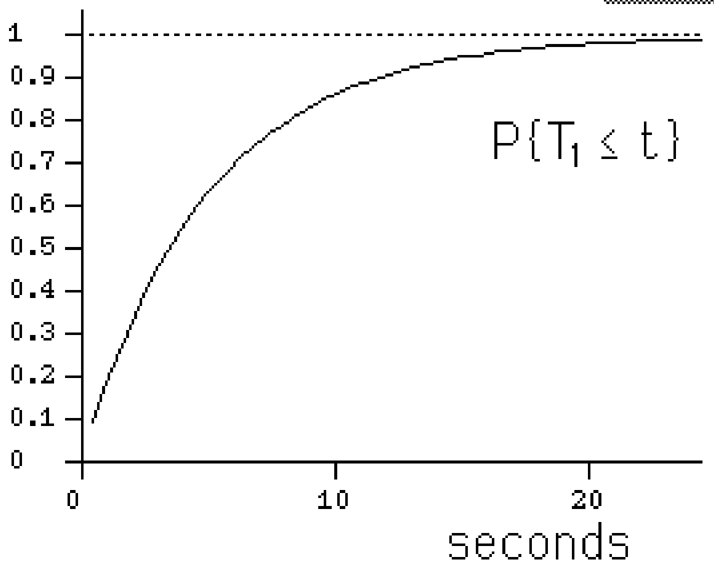


Exponential Distribution

$\lambda = 0.2/\text{sec.}$

$$F(t) = 1 - e^{-\lambda t}$$

t	F(t)
1	0.18127
2	0.32968
3	0.45119
4	0.55067
5	0.63212
6	0.69881
7	0.75340
8	0.79810
9	0.83470
10	0.86466
11	0.88920
12	0.90928
13	0.92573
14	0.93919
15	0.95021
16	0.95924
17	0.96663
18	0.97268
19	0.97763
20	0.98168



Memoryless Property

Exponential Distribution

Suppose that it is known that, at time t_0, the first arrival has not yet occurred, i.e., $T_1 > t_0$.

What is the conditional distribution of T_1 ?

That is, what is $P\{T_1 \le t \mid T_1 > t_0\}$ for $t > t_0$?

Memoryless Property

Exponential Distribution

$$P\{T_1 \leq t \mid T_1 > t_0\} = \frac{P\{T_1 \leq t \cap T_1 > t_0\}}{P\{T_1 > t_0\}} = \frac{P\{t_0 \leq T_1 \leq t\}}{P\{T_1 > t_0\}}$$

$$= \frac{F(t) - F(t_0)}{1 - F(t_0)} = \frac{(1 - e^{-\lambda t}) - (1 - e^{-\lambda t_0})}{e^{-\lambda t_0}}$$

$$= \frac{e^{-\lambda t_0} - e^{-\lambda t}}{e^{-\lambda t_0}} = 1 - e^{-\lambda(t - t_0)}$$

Memoryless Property

Exponential Distribution

$$P\{T_1 \le t \mid T_1 > t_0\} = 1 - e^{-\lambda(t-t_0)} = P\{T_1 \le t - t_0\}$$

If the time τ is reckoned from time t_0, i.e., $\tau = t - t_0$, then

$$P\{T_1 \le t \mid T_1 > t_0\} = P\{T_1 \le t - t_0\} = P\{T_1 \le \tau\}$$

In other words, the failure of an arrival to occur before time t_0 does not alter one's prediction of the length of time (from t_0) before the next arrival.

Time of k^{th} Arrival

Let T_k = time of k^{th} arrival,

$\tau_k = T_k - T_{k-1}$ = time between arrivals k-1 and k.

Suppose that τ_k (k=1,2,3,....) have identical and independent exponential distributions with rate .

Then T_k is the *sum* of k random variables with exponential distributions.

It is said to have a *k-Erlang* distribution.

Erlang Distribution

time of k^{th} arrival in a Poisson process

Density function

$(k>0,\ \lambda>0,\ t\geq 0)$

$$f(t) = \frac{\lambda(\lambda t)^{k-1}e^{-\lambda t}}{(k-1)!}$$

$$= \frac{\lambda(\lambda t)^{k-1}e^{-\lambda t}}{\Gamma(k)}$$

where the ***Gamma*** function is defined by

(for k>0, not necessarily integer!)

$$\Gamma(k) = \int_0^{\infty} e^{-u}u^{k-1}\,du$$

$= \mathbf{(k\text{-}1)!}$ if $\mathbf{k}$ integer

Erlang Distribution

CDF

$$F(t) = \frac{\Gamma(k, \lambda t)}{\Gamma(k)}$$

where $\Gamma(k, x)$ is the *"incomplete Gamma function"* defined by

$$\Gamma(k, x) = \int_0^x e^{-u} u^{k-1}\, du$$

tabulated

Alternate computation, when k is integer

CDF

$$F(t) = P\{T_k \le t\} = P\{N_t \ge k\}$$
$$= 1 - P\{N_t < k\}$$
$$= 1 - P\{N_t \le k-1\}$$

where N_t = # arrivals at time t

has the *Poisson* distribution:

$$F(t) = 1 - \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

Erlang Distribution

Mean Value $\mu = \dfrac{k}{\lambda}$

Variance $\sigma^2 = \dfrac{k}{\lambda^2}$

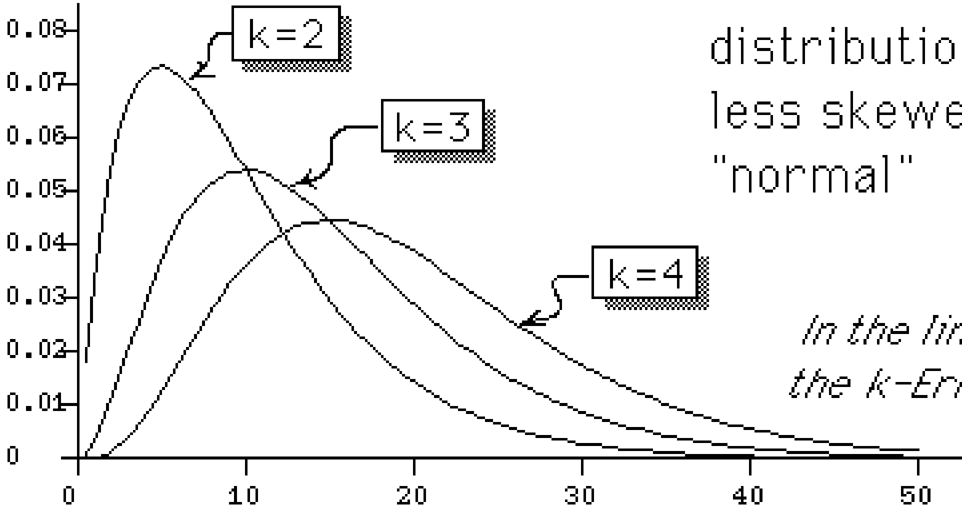
(These expressions result from the fact that the random variable is the sum of k i.i.d. random variables.)

More generally, when k is not an integer, the probability distribution is called the ***Gamma*** distribution.

Erlang Distribution

example:

$\lambda = 0.2$



As k increases, the distribution becomes less skewed, and more "normal"

In the limit, as $k \to \infty$, the k-Erlang distribution converges to the Normal distribution!

Bernouilli process		Poisson process
Binomial distn.	# of events	Poisson distn.
Geometric distn.	time until 1st event	Exponential distn.
Pascal distn.	time until k^{th} event	Erlang distn.