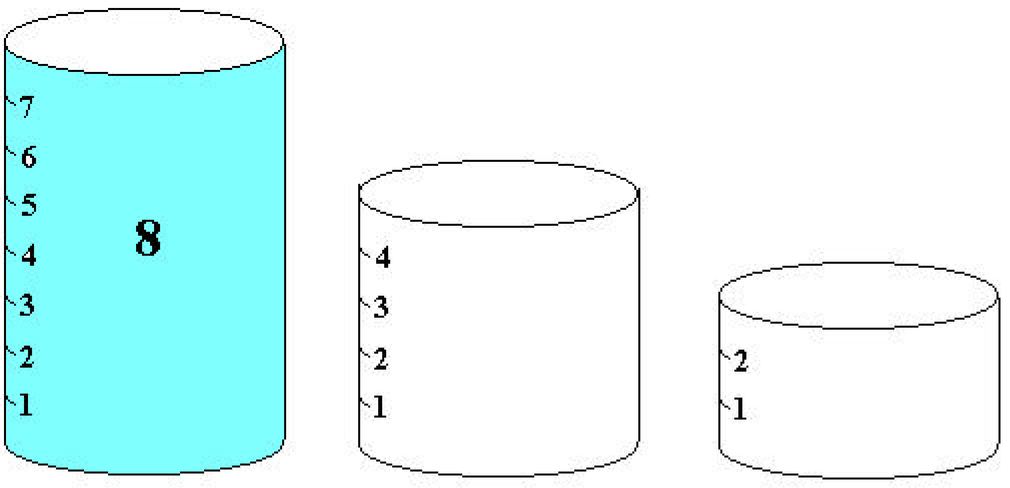
Milkman's Problem
A Graph Theoretical Model

The problem

A milkman has three containers of capacities 8 gallons, 5 gallons, and 3 gallons. The 8-gallon container is full of milk. How can he divide the milk into two 4-gallon portions without using anything but his three containers?



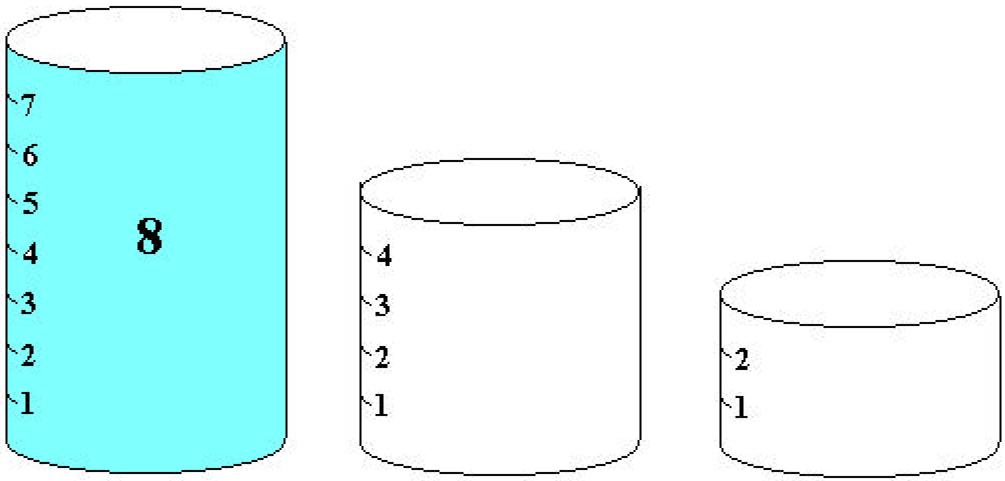
Define the *state* of the system to be (x,y,z) where

x= # gallons of milk in 8-gallon container

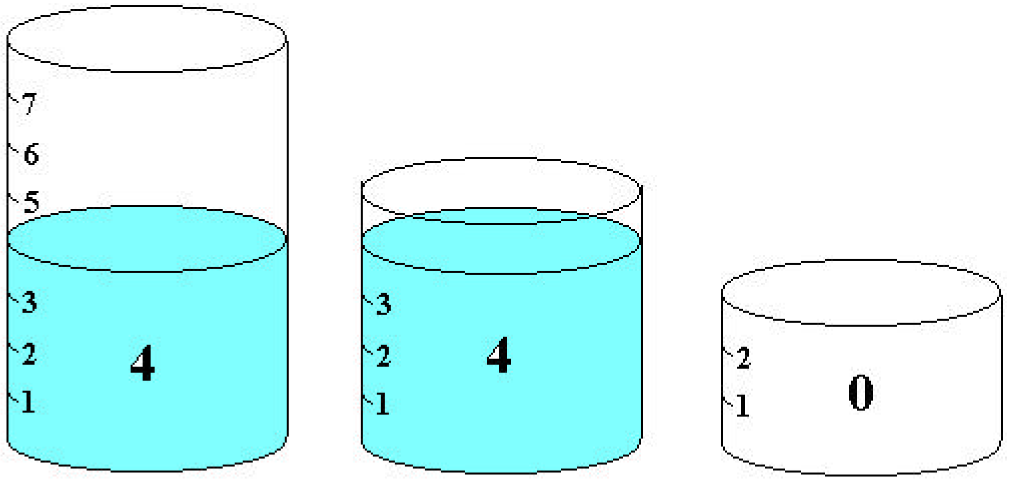
y= # gallons of milk in 5-gallon container

z= # gallons of milk in 3-gallon container

The initial state, then, is (8,0,0)



The desired state is (4,4,0)



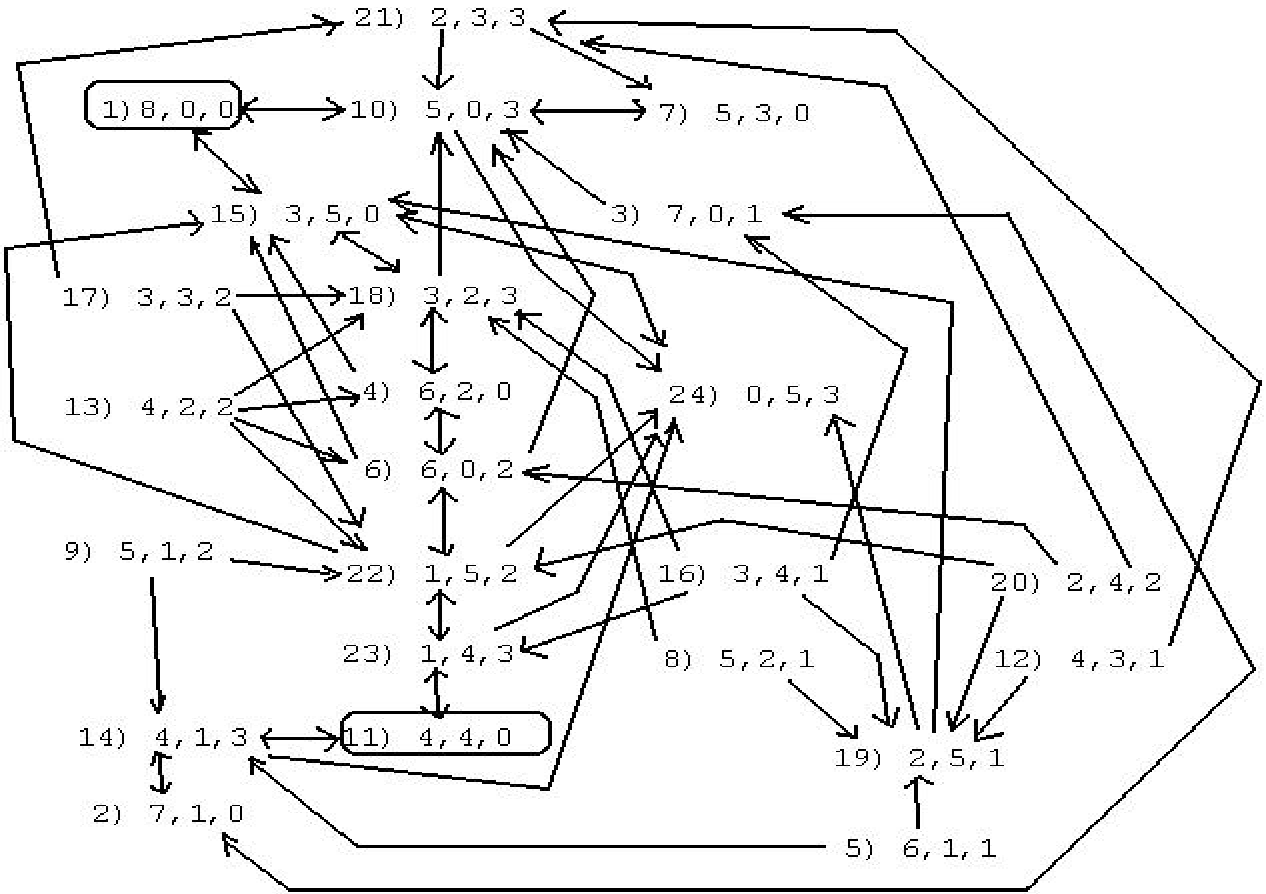
What are the intermediate states to get from (8,0,0) to (4,4,0)?

Possible states are

#	State	#	State	#	State
1	(8,0,00	9	(5,1,2)	17	(3,3,2)
2	(7,1,0)	10	(5,0,3)	18	(3,2,3)
3	(7,0,1)	11	(4,4,0)	19	(2,5,1)
4	(6,2,0)	12	(4,3,1)	20	(2,4,2)
5	(6,1,1)	13	(4,2,2)	21	(2,3,3)
6	(6,0,2)	14	(4,1,3)	22	(1,5,2)
7	(5,3,0)	15	(3,5,0)	23	(1,4,3)
8	(5,2,1)	16	(3,4,1)	24	(0,5,3)

ADJACENCY

		800	710	701	620	611	602	530	521	512	503	440	431	422	413	350	341	332	323	251	242	233	152	143	053
1]	8 0 0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
2]	7 1 0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
3]	7 0 1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4]	6 2 0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
5]	6 1 1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
6]	6 0 2	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	0	0
7]	5 3 0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8]	5 2 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
9]	5 1 2	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
10]	5 0 3	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
11]	4 4 0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
12]	4 3 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
13]	4 2 2	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
14]	4 1 3	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
15]	3 5 0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
16]	3 4 1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0
17]	3 3 2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0
18]	3 2 3	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
19]	2 5 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
20]	2 4 2	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0
21]	2 3 3	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22]	1 5 2	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1
23]	1 4 3	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	1
24]	0 5 3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0


21) 2,3,3
1) 8,0,0
10) 5,0,3
7) 5,3,0
15) 3,5,0
3) 7,0,1
17) 3,3,2
18) 3,2,3
13) 4,2,2
4) 6,2,0
24) 0,5,3
6) 6,0,2
9) 5,1,2
22) 1,5,2
16) 3,4,1
20) 2,4,2
23) 1,4,3
8) 5,2,1
12) 4,3,1
14) 4,1,3
11) 4,4,0
19) 2,5,1
2) 7,1,0
5) 6,1,1

REACHABILITY

	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4
1	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
2	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
3	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
4	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
5	1	1	1	1	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1
6	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
7	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
8	1	1	1	1	0	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1
9	1	1	1	1	0	1	1	0	1	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
10	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
11	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
12	1	1	1	1	0	1	1	0	0	1	1	1	0	1	1	0	0	1	1	0	1	1	1	1
13	1	1	1	1	0	1	1	0	0	1	1	0	1	1	1	0	0	1	0	0	0	1	1	1
14	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
15	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
16	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	1	0	1	1	0	0	1	1	1
17	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	1	1	0	0	1	1	1	1
18	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
19	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1
20	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	1	1	1
21	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	1	1	1	1
22	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
23	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1
24	1	1	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0	0	1	1	1

Row #1 of the first ten powers of the adjacency matrix

n	1	2	3	4	6	7	10	11	14	15	18	22	23	24
1	0	0	0	0	0	0	1	0	0	1	0	0	0	0
2	2	0	0	0	0	1	0	0	0	0	1	0	0	2
3	0	0	0	1	0	0	4	0	0	5	0	0	0	0
4	9	0	0	0	1	4	0	0	0	1	6	0	0	9
5	1	0	0	7	0	0	20	0	0	25	1	1	0	1
6	45	0	0	1	8	20	2	0	0	11	32	0	1	46
7	13	0	0	40	1	2	105	**1**	0	132	12	9	0	14
8	237	0	0	13	49	105	28	0	1	89	172	1	10	246
9	117	1	0	221	14	28	563	11	0	718	102	59	1	129
10	1281	0	0	116	280	563	261	1	12	642	939	15	70	1341

This indicates that there is one path from node # 1 to node # 11, i.e., (8,0,0) $\rightarrow$ (4,4,0), of length 7 edges.

Shortest Paths Originating at Node #1

From	Length	Predecessor
1	0	0
2	9	14
3	10	2
4	3	18
6	4	4
7	2	10
10	1	1
11	7	23
14	8	11
15	1	1
18	2	15
22	5	6
23	6	22
24	2	10

From	Length	Predecessor
1	0	0
2	9	14
3	10	2
4	3	18
6	4	4
7	2	10
10	1	1
11	7	23
14	8	11
15	1	1
18	2	15
22	5	6
23	6	22
24	2	10

That is, on the path originating at node #1, the predecessor of node #11 is node #23.

(1,4,3) → (4,4,0)

The predecessor of node #23 is node #22

(1,5,2) → (1,4,3)

The predecessor of node #22 is node #6

(6,0,2) → (1,5,2)

etc.

Tracing through the predecessor list, we find that the path from state 1 to state 11 is:

1®15®18®4®6®22®23®11

That is,

8,0,0®3,5,0	Fill #2 from #1
3,5,0®3,2,3	Fill #3 from #1
3,2,3®6,2,0	Empty #3 into #1
6,2,0®6,0,2	Empty #2 into #3
6,0,2®1,5,2	Fill #2 from #1
1,5,2®1,4,3	Fill #3 from #2
1,4,3®4,4,0	Empty #3 into #1

a. Represent each state by a node of a graph, with edges linking states which can be obtained by pouring milk from one container to another. For example, from the initial state of the system, #1, i.e., (8,0,0), states #10 and 15 can be obtained by a single operation.
b. Is state #11 reachable from state #1? If so, by what path?
c. Give instructions to the milkman which explain the steps which he must perform to solve his problem.
d. Is there a state which is *not* reachable from #1?