Facility Location Problem in a Network

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Median Problem
minimizing the sum of weighted shortest path lengths

Center Problem
minimizing the maximum of (possibly) weighted shortest path lengths

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The $p$-Median Problem

Given a network with nodes $j = 1, 2, \ldots, n$
where $w_j =$ "weight" of node $j$
(e.g., volume of shipments)
Let $d(X, j) =$ distance from node $j$ to
the nearest point in the set $X$
Find $X = \{x_1, x_2, \ldots, x_p\}$ which

minimizes $\tau(X) = \sum_{j=1}^{n} w_j d(X, j)$

The points in $X$ are called $p$-medians.

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Hakimi’s Theorem

At least one set of p-mediians exist solely on the nodes of the network.

That is, we need search only among the nodes for the p-mediians!
Where should a single facility be located to serve the eight cities?

Objective:
Minimize the sum of the distances to the cities weighted by their demands.
Find the matrix of shortest path lengths between nodes of the network (e.g., by Floyd's algorithm)
### Table 1: Shortest Paths

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
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<tbody>
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<td>1</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

**Weighted Shortest Paths**

$$\sum_{j} W_{jd_{ij}}$$

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The optimal location for a single facility to serve the 8 cities is at city C.

What if two facilities were to be used?
Consider all pairs of potential facility sites:

Examples: \( \sqrt{\text{select minimum shipping cost in each column}} \)

\[
\begin{array}{c|cccccc}
A & 0 & 4 & 6 & 6 & 6 & 0 & 32 & 6 \\
B & 12 & 0 & 4 & 9 & 4 & 0 & 0 & 28 & 5 \\
\end{array}
\]

\[ 47 = \sum_{j} \min_{i=A,B} \{W_{ij}d_{ij}\} \]

\[
\begin{array}{c|cccccc}
D & 6 & 3 & 2 & 0 & 4 & 0 & 24 & 4 \\
E & 18 & 4 & 6 & 12 & 0 & 0 & 32 & 6 \\
\end{array}
\]

\[ 39 = \sum_{j} \min_{i=D,E} \{W_{ij}d_{ij}\} \]

\[
\begin{array}{c|cccccc}
A & 0 & 4 & 6 & 6 & 6 & 0 & 32 & 6 \\
G & 24 & 7 & 10 & 18 & 8 & 0 & 0 & 3 \\
\end{array}
\]

\[ 25 = \sum_{j} \min_{i=A,G} \{W_{ij}d_{ij}\} \]

There are \( \binom{8}{2} = 28 \) such combinations to evaluate!
How might one find the 3-median set?

Requires considering \( \binom{8}{3} = 56 \) combinations!

\[
\begin{array}{cccccc}
A & 0 & 4 & 6 & 6 & 6 & 0 & 32 & 6 \\
B & 12 & 0 & 4 & 9 & 4 & 0 & 28 & 5 \\
C & 9 & 2 & 0 & 3 & 0 & 3 & 0 & 20 & 3 \\
D & 6 & 3 & 2 & 0 & 4 & 0 & 24 & 4 \\
\end{array}
\]

\[
29 = \sum_{j} \text{minimum}\{W_{j}d_{ij}\} \quad i = A, B, C
\]

\[
34 = \sum_{j} \text{minimum}\{W_{j}d_{ij}\} \quad i = A, B, D
\]

e tc.

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APL evaluation of $\sum_j \min_{i \in S} \{W_{jd_{ij}}\}$

$+/l \neq (D \times (\rho D) \rho W)[S]$
Math Programming Model of the \( p \)-Median Problem

Variables

\[ X_{ij} = \text{fraction of demand of customer } j \text{ supplied by facility at location } i \]

\[ Y_i = \begin{cases} 
1 & \text{if a facility is located at site } i \\
0 & \text{otherwise} 
\end{cases} \]

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Math Programming Model of the p-Median Problem

Min \[ \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} D_{ij} X_{ij} \]

subject to \[ \sum_{i=1}^{m} X_{ij} = 1 \quad \forall j = 1, \ldots, n \]

\[ X_{ij} \leq Y_i \quad \forall i = 1, \ldots, m; j = 1, \ldots, n \]

\[ \sum_{i=1}^{m} Y_i = p \]

\[ X_{ij} \geq 0 \quad \forall i = 1, \ldots, m; j = 1, \ldots, n \]

\[ Y_i \in \{0, 1\} \quad \forall i = 1, \ldots, m \]
Heuristic Algorithm for the p-Median Problem

1. Initialization:
   Let k=1. Find the 1-median (the set S=X_1)

2. Facility Addition:
   Evaluate the (n-k) combinations of S with a node r not in S, i.e.,
   \[ \sum_{j \in S \cup \{r\}} \min \{W_{jd_{ij}}\} \quad \forall r \not\in S \]
   Add to S the node yielding the lowest objective function and set k=k+1.

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3. Facility Substitution:
Evaluate each of the $k \times (n-k)$ sets obtained by substituting a node not in $S$ for a node in $S$, i.e.

$$\sum_{j} \min_{i \in S \cup \{r\} \setminus \{s\}} \{W_{jd_{ij}}\} \quad \forall \ r \notin S \& s \in S$$

Replace $S$ by the best set evaluated.

4. If $S$ contains $p$ nodes, i.e., $k=p$, STOP.
   Otherwise, return to step 2.
K-median Facility Location Problem

1-Median the one-median

Cost = 40

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... beginning with 1-median set \( \{C\} \)

**2-Median**

Trial additions:

- Add 1 2 4 5 6 7 8
- Cost 31 38 34 37 30 20 29

Addition result: Locations 3 7
Cost: 20

Add facility at node G to the set.
Substitution Step

<table>
<thead>
<tr>
<th>Cost</th>
<th>Locations</th>
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<tbody>
<tr>
<td>25</td>
<td>1 7</td>
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<td>31</td>
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<td>29</td>
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</table>

Substitution result: Locations 4 7
Cost: 18

substitute D for C in the set
... begin with $D$ & $G$ in set

3-Median

Trial additions:
Add 1 2 3 5 6 8
Cost 12 15 14 14 16 15
Addition result: Locations 4 7 1
Cost: 12
### Substitution Step

<table>
<thead>
<tr>
<th>Cost</th>
<th>Locations</th>
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Substitution result: Locations 371
Cost: 11

substitute C for D in the set
... begin with $A$, $C$, & $G$ in set

4-Median

<table>
<thead>
<tr>
<th>Trial additions:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Add $2 4 5 6 8$</td>
</tr>
<tr>
<td>cost $9 8 8 9 8$</td>
</tr>
</tbody>
</table>

Add $D$ to the set

Addition result: Locations $3 7 1 4$
Cost: 8
begin with A, C, D, & G (1,3,4,7)

Substitution Step

<table>
<thead>
<tr>
<th>Cost</th>
<th>Locations</th>
</tr>
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<tbody>
<tr>
<td>9</td>
<td>2 7 1 4</td>
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<tr>
<td>26</td>
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</tr>
<tr>
<td>8</td>
<td>3 7 1 8</td>
</tr>
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</table>

Substitution result: Locations 5 7 1 4
Cost: 8
Allocation of Customers to Warehouses

<table>
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<th></th>
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<th>B</th>
<th>C</th>
<th>D</th>
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<tr>
<td>G</td>
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<tr>
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<td>7</td>
<td>10</td>
<td>18</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

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1-Median of a Tree

For any set \( C \) of vertices, define \( W(C) = \sum_{i \in C} w_i \)

**Theorem** Let \([a,b]\) be any edge of a tree, and let \( A = \) set of vertices reachable from \( a \) without passing through \( b \)

\( B = \) set of vertices reachable from \( b \) without passing through \( a \).

Then \( W(A) \geq W(B) \) implies \( \tau(a) \leq \tau(b) \)
$W(A) \geq W(B)$ implies $\tau(a) \leq \tau(b)$
To find the 1-median of a tree:

0. Let \( w = \sum_{i \in N} w_i \). Select any vertex \( j \).

1. If \( w_j \geq \frac{1}{2} w \), then stop; \( j \) is a 1-median.

2. If \( j \) has degree 1, let \( k \) be its neighbor, i.e., \([k,j]\) will be an edge. Replace \( w_k \) with \( w_k + w_j \), and delete vertex \( j \) from the tree.

Else find an elementary chain from vertex \( j \) to a vertex \( k \) with degree 1 (preferably using previously unused edges.)

Let \( j = k \) and return to step 1.
Example

Find the 1-median of the tree:

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Let's choose to begin with vertex #7.
Total "demand" $w = \sum_{i \in N} w_i$ is 30.
\[ w_7 < \frac{W}{2} = 15. \] Select neighbor (vertex \#6), and replace \( w_6 \) with \( w_6 + w_7 = 3 \). Delete vertex \#7.
\[ w_6 < \frac{15}{2} = 15. \] Find a path 6→8 to a vertex (#8) with degree 1:
$w_8 < \frac{W}{2} = 15$. Select neighbor (vertex \#6), update $w_6$, and delete vertex \#8:
\( w_6 \leq \frac{W}{2} = 15 \). Select neighbor (vertex \#5), update \( w_5 \), and delete vertex \#6:
\[ w_{5} < \frac{W}{2} = 15. \] Select chain 5\( \rightarrow \)9\( \rightarrow \)10\( \rightarrow \)12 to vertex #12, which has degree 1.
$w_{12} < W/2 = 15$. Select neighbor (vertex #10), update $w_{10}$, and delete vertex #12.
... after several more iterations, the tree is as shown, where vertex #9 is being considered.
$w_9 < \frac{\bar{w}}{2} = 15$, so we select its neighbor (vertex #5), update $w_5$, and delete vertex #9.
\[ w_5 > \bar{w}/2 = 15, \text{ so we stop; Vertex } \#5 \text{ is the 1-median.} \]
$W(A) \geq W(B)$ implies $\tau(a) \leq \tau(b)$

Edge (5,9): $W(9)=14 < 16 = W(5)$ implies $\tau(9) > \tau(5)$
W(A) ≥ W(B) implies τ(a) ≤ τ(b)

Edge [5, 4]: W(5) = 22 > 8 = W(4) implies τ(4) > τ(5)