Classification of States of a Markov chain
A state \( i \) is \textit{recurrent} if, given that the Markov chain starts in state \( i \), the probability that it eventually returns to state \( i \) is one.

i.e., \[ \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1 \]

\( f_{ij}^{(n)} \) = Probability that the first visit to state \( j \) occurs at stage \( n \), given that the initial state is \( i \).

A state which is not recurrent is said to be \textit{transient}.
State $j$ is reachable from state $i$

\[ i \rightarrow j \]

States $i$ & $j$

\[ i \leftrightarrow j \]

If state $i$ is recurrent, and states $i$ & $j$ communicate, then state $j$ is recurrent.
The *period* $d(i)$ of state $i$ is the greatest common divisor of all the integers $n \geq 1$ for which

$$\rho_{ii}^{(n)} > 0$$

**Examples**

- $d(1) = d(2) = 2$
- $d(1) = d(2) = 1$
- $d(1) = d(2) = d(3) = 2$

If $i \leftrightarrow j$, then $d(i) = d(j)$.

A Markov chain with $d(i) = 1$ for all $i$ is called *aperiodic*
A set of states is *closed* if no state not in the set is reachable from a state in the set.

A *minimal closed set* is a closed set which has no closed proper subsets.

The closed sets are:

- \( \{1, 2, 3, 4, 5, 6, 7\} \)
- \( \{1, 2, 3\} \)
- \( \{1, 2, 3, 4, 6, 7\} \)
- \( \{7\} \)

Both these closed sets are minimal.
A minimal closed set is said to be *irreducible*.

A Markov chain is called *irreducible* if the set of its states is a minimal closed set.

(A Markov chain is *irreducible* if and only if every pair of its states communicate.)
A state which forms a closed set, i.e., which cannot reach another state, is said to be **absorbing**.

If state $j$ is absorbing, then

$$p_{jj} = p_{jj}^{(n)} = 1$$

for all $n=1, 2, ...$
In a Markov chain with finitely many states, a member of a minimal closed set is *recurrent* and other states are *transient*.

States 1, 2, 3, & 7 are recurrent.
If state \( j \) is recurrent, but

\[
\lim_{n \to \infty} p_{ij}^{(n)} = 0 \quad \text{for any state } i,
\]

then state \( j \) is said to be \textit{null}.

An irreducible Markov chain with \textit{finitely} many states has

- \textit{no} recurrent null states
- \textit{no} transient states
Absorption Analysis

Consider a Markov chain with $N$ states:

- $r$ absorbing states
- $s = N - r$ transient states

Partition the transition probability matrix $P$:

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \begin{cases} r \text{ rows} \\ r \text{ rows} \end{cases} \begin{cases} \begin{array}{c} \begin{array}{c} \text{columns} \end{array} \\ s \text{ columns} \end{array} \end{cases}$$
The Powers of $P$:

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$$

$$P^2 = \begin{bmatrix} I & 0 \\ R+QR & Q^2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} I & 0 \\ R+QR+Q^2R & Q^3 \end{bmatrix}$$

$$P^n = \begin{bmatrix} I & 0 \\ \left(I+Q+Q^2+\ldots+Q^{n-1}\right)R & Q^n \end{bmatrix} \begin{cases} \text{absorbing} \\ \text{transient} \end{cases}$$
Let states $i$ and $j$ both be transient, and define

$$e_{ij} = \text{expected \# of visits to state } j, \text{ given that the system begins in state } i$$

$$e_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

and the $r \times r$ matrix:

$$E = \sum_{n=0}^{\infty} Q^n = (I - Q)^{-1}$$

since

$$(I - Q)(I + Q + Q^2 + \ldots) = I + Q - Q + Q^2 - Q^2 + \ldots = I$$
Absorption probability

Let state $i$ be transient and state $j$ absorbing, and define:

$$a_{ij} = \text{probability that the system enters the absorbing state } j \text{ at some future time, given that it is initially in transient state } i$$

$$a_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

absorption probability (an infinite sum)
An alternate method for computing these probabilities:

Condition on the state entered at stage #1:

\[ a_{ij} = \sum_{k=1}^{N} P\{\text{system enters state } j \mid X_1 = k\} \cdot P\{X_1 = k\} \]

\[ = P\{\text{system enters state } j \mid X_1 = j\} \cdot P\{X_1 = j\} \]

\[ + \sum_{k \text{ absorbing, } k \neq j} P\{\text{system enters state } j \mid X_1 = k\} \cdot P\{X_1 = k\} \]

\[ + \sum_{k \text{ transient}} P\{\text{system enters state } j \mid X_1 = k\} \cdot P\{X_1 = k\} \]

\[ = 1 p_{ij} + 0 + \sum_{k=1}^{s} a_{kj} p_{ik} \]

\[ a_{ij} = p_{ij} + \sum_{k=1}^{s} a_{kj} p_{ik} \]
\[ a_{ij} = p_{ij} + \sum_{k=1}^{N} a_{kj} p_{ik} \]  \quad \text{, \(i\) transient, \(j\) absorbing}

\textit{In matrix form:}

\[ A = R + QA \]
\[ A - QA = R \]
\[ (I - Q)A = R \]
\[ A = (I - Q)^{-1}R \]

\[ A = ER \]