Lagrangian Duality

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Consider the inequality-constrained problem:

\[
\begin{align*}
\text{Minimize } & \quad f(x) \\
\text{subject to } & \quad g_i(x) \leq 0, \ i = 1, 2, \ldots, m \\
& \quad x \in X
\end{align*}
\]

Define the Lagrangian function:

\[
L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)
\]
Based upon this Lagrangian function, we define two functions:

\[ L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \]

**Primal Objective**

\[ \bar{L}(x) \equiv \text{Maximum} \ L(x, \lambda) \quad \lambda \geq 0 \]

*Fix "x" and maximize with respect to the Lagrange multiplier*

**Dual Objective**

\[ \hat{L}(\lambda) \equiv \text{Minimum} \ L(x, \lambda) \quad x \in X \]

*Fix the Lagrange multiplier and minimize w.r.t. "x"*
\[ L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \]

\[ \bar{L}(x) \equiv \text{Maximum } L(x, \lambda) \quad \lambda \geq 0 \]

\[ \hat{L}(\lambda) \equiv \text{Minimum } L(x, \lambda) \quad x \in X \]

**Weak Duality Relationship:** for all \( x \in X \) and \( \lambda \geq 0 \),

\[ \text{Maximum } L(x, \lambda) \equiv \bar{L}(x) \geq L(x, \lambda) \geq \hat{L}() \equiv \text{Minimum } L(x, \lambda) \quad x \in X \]

**primal objective**

**dual objective**

\( \odot D.L. \text{Bricker, U.of IA, 1998} \)
\[
L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)
\]

Primal Objective

\[
\overline{L}(x) \equiv \max_{\lambda \geq 0} L(x, \lambda)
\]

\[
= \begin{cases} 
  f(x) & \text{if } g_i(x) \leq 0 \quad \forall i \\
  +\infty & \text{if } g_i(x) > 0 \text{ for some } i
\end{cases}
\]

If \( g_i(x) \leq 0 \quad \forall i \) then optimal \( \lambda_i \)'s are zero;

otherwise, if \( g_i(x) > 0 \) for some \( i \), \( L(x, \lambda) \) is unbounded above as \( \lambda_i \to +\infty \).
\[ L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \]

**Primal Problem**

Minimize \( \bar{L}(x) \) \( \quad x \in X \)

**Dual Problem**

Maximize \( \hat{L}(\lambda) \) \( \lambda \geq 0 \)

\( \bar{L}(x) \equiv \max_{\lambda \geq 0} L(x, \lambda) \)

\( \hat{L}(\lambda) \equiv \min_{x \in X} L(x, \lambda) \)
Primal Problem

\[
\begin{align*}
\text{Minimize } & \bar{L}(x) \\
\text{subject to } & x \in X
\end{align*}
\]

where

\[
\bar{L}(x) = \begin{cases} f(x) & \text{if } g_i(x) \leq 0 \ \forall \ i \\ +\infty & \text{if } g_i(x) > 0 \text{ for some } i \end{cases}
\]

If there exists an \( x \) feasible in \( \{ g_i(x) \leq 0 \ \forall \ i \} \), then we can restrict our search for the minimizing \( x \) to such \( x \)'s, and therefore

\[
\text{Minimum } \bar{L}(x) = \text{Minimum } \{ f(x) \mid g_i(x) \leq 0 \ \forall \ i \}
\]
And so we see that

Primal Problem

\[
\begin{array}{l}
\text{Minimize } L(x) \\
\quad x \in X
\end{array}
\]

\quad \text{is equivalent to our original problem:}

\[
\begin{array}{l}
\text{Minimize } f(x) \\
\text{subject to } \\
\quad g_i(x) \leq 0, \ i = 1, 2, \ldots, m \\
\quad x \in X
\end{array}
\]

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Weak Duality Relationship

For all $x \in X$ and $\lambda \geq 0$, 

$$\bar{L}(x) \geq L(x, \lambda) \geq \hat{L}(\lambda)$$

In particular, if $x^*$ and $\lambda^*$ are the primal and dual optima, respectively, then

$$\bar{L}(x^*) \geq \hat{L}(\lambda^*)$$

i.e.,

$$\bar{L}(x^*) - \hat{L}(\lambda^*) \geq 0$$

Duality Gap

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Weak Duality Relationship

For all $x \in X$ and $\lambda \geq 0$,

$$\overline{L}(x) \geq L(x, \lambda) \geq \hat{L}(\lambda)$$

That is, any feasible dual solution gives a lower bound on all primal solutions, including of course the optimal.... this property is often used to advantage in branch-and-bound algorithms for combinatorial problems.
Definition \((\bar{x}, \bar{\lambda})\) is a saddlepoint of \(L(x, \lambda)\)

if \(L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \forall x \in X\)

(which implies that \(\hat{L}(\bar{x}) = L(\bar{x}, \bar{\lambda})\))

and \(L(\bar{x}, \bar{\lambda}) \geq L(\bar{x}, \lambda) \forall \lambda \geq 0\)

(which implies that \(\hat{L}(\bar{\lambda}) = L(\bar{x}, \bar{\lambda})\))
If \((\bar{x}, \bar{\lambda})\) is a saddlepoint of \(L(x, \lambda)\)

then

\[
\bar{L}(\bar{x}) = \bar{L}(\bar{x}, \bar{\lambda}) = \bar{L}(\bar{\lambda})
\]

so that the duality gap is zero!
**EXAMPLE**

Minimize $4x_1^2 + 2x_1x_2 + x_2^2$

subject to $3x_1 + x_2 \geq 6$

$x_1 \geq 0, x_2 \geq 0$

Define: $g(x) = 6 - 3x_1 - x_2$

$X = \{(x_1,x_2) \mid x_1 \geq 0, x_2 \geq 0 \}$

The Lagrangian is

$L(x, \lambda) = 4x_1^2 + 2x_1x_2 + x_2^2 + \lambda (6 - 3x_1 - x_2)$

Dual objective:

$L(\lambda) = \min_{x \geq 0} \{4x_1^2 + 2x_1x_2 + x_2^2 + \lambda (6 - 3x_1 - x_2) \}$
The K-K-T necessary conditions for optimality of \( x_1, x_2 \geq 0 \) are:

(for \( \lambda \) fixed)

\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 8x_1 + 2x_2 - 3\lambda \geq 0 \\
\frac{\partial L}{\partial x_2} &= 2x_1 + 2x_2 - \lambda \geq 0 \\
x_1 \left[ \frac{\partial L}{\partial x_1} \right] &= 0, \quad x_2 \left[ \frac{\partial L}{\partial x_2} \right] = 0
\end{align*}
\]

with solution:

\[
x_1^*(\lambda) = \frac{\lambda}{3}, \quad x_2^*(\lambda) = \frac{\lambda}{6}
\]

\( x_1, x_2 \geq 0 \ \forall \ \lambda \geq 0 \)
And so the dual objective is

$$\tilde{L}(\lambda) = L\left(\frac{\lambda}{3}, \frac{\lambda}{6}, \lambda \right)$$

$$= 6 \lambda - \frac{7}{12} \lambda^2 \quad \leftarrow \text{a CONCAVE function of } \lambda$$

and the dual problem is

Maximize $6 \lambda - \frac{7}{12} \lambda^2$

subject to $\lambda \geq 0$
Maximize \[6 \lambda - \frac{7}{12} \lambda^2\]
subject to \(\lambda \geq 0\)
Dual problem:

Maximize $6 \lambda - \frac{7}{12} \lambda^2$
subject to $\lambda \geq 0$

The necessary (and sufficient) conditions for optimality are

\[
\frac{d\tilde{L}(\lambda)}{d\lambda} = 6 - 2 \left(\frac{7}{12}\right) \lambda \leq 0, \quad \lambda \left[\frac{d\tilde{L}(\lambda)}{d\lambda}\right] = 0
\]

\[\Rightarrow \quad \lambda^* = \frac{36}{7} \quad \tilde{L}(\lambda^*) = \tilde{L}\left(\frac{36}{7}\right) = \frac{108}{7}\]
The corresponding values of $x^*$ which optimize the Lagrangian subproblem, i.e., the problem of evaluating the dual objective $\tilde{L}$, are:

$$
\begin{align*}
    x_1^*(\lambda^*) &= \lambda^*/3 = \frac{36}{7}/3 = \frac{12}{7}, \\
    x_2^*(\lambda^*) &= \lambda^*/6 = \frac{36}{7}/6 = \frac{6}{7}
\end{align*}
$$

at which the primal objective, $4x_1^2 + 2x_1x_2 + x_2^2$, also has the value $\frac{108}{7}$.
\[ L(x) = \begin{cases} 
4x_1^2 + 2x_1x_2 + x_2^2 & \text{if } 3x_1 + x_2 \leq 6, \ x \geq 0 \\
+ \infty & \text{otherwise}
\end{cases} \]

\[ x_1^* = \frac{12}{7}, \quad x_2^* = \frac{6}{7}, \quad \overline{L}(x^*) = \frac{108}{7} \]

\[ \overline{L}(\lambda) = 6 \lambda - \frac{7}{12} \lambda^2, \quad \lambda \geq 0 \]

\[ \lambda^* = \frac{36}{7}, \quad \overline{L}(\lambda^*) = \frac{108}{7} \]

\[ \overline{L}(x^*) = \overline{L}(\lambda^*) \]

No Duality Gap!
Geometric Interpretation

Define \( G \equiv \{ (z_1, z_2) \mid z_1 = g(x), z_2 = f(x) \text{ for } x \in X \} \)

Primal can be restated as:

Minimize \( z_2 \)
subject to
\( z_1 \leq 0, \quad z \in G \)

Minimize \( f(x) \)
subject to \( g(x) \leq 0, \quad x \in X \)
For fixed $\lambda$, 

$$\tilde{L}(\lambda) = \min_{z \in G} \{ z_2 + \lambda z_1 \}$$

$\tilde{L}(\lambda)$ is the $z_2$-intercept of the supporting hyperplane of $G$ with slope $-\lambda$.

Dual problem is to find support with maximum $z_2$-intercept.
The optimal $\lambda$ is that for which the $z$-intercept of the supporting hyperplane (in this case, line) is maximized.
When $G$ is nonconvex, a duality gap is possible!
EXAMPLE  

integer linear program

\[
\begin{align*}
\text{Minimize} & \quad 3x_1 + 7x_2 + 10x_3 \\
\text{subject to} & \quad x_1 + 3x_2 + 5x_3 \geq 7 \\
& \quad x_j \in \{0,1\} , \ j=1,2,3
\end{align*}
\]

Define:

\[
X \equiv \{ \ x = (x_1,x_2,x_3) \ | \ x_j \in \{0,1\} \} \\
= \{0,1\} \times \{0,1\} \times \{0,1\} \quad \text{Cartesian product}
\]

\[
g(x) \equiv 7 - x_1 - 3x_2 - 5x_3
\]

Lagrangian function:

\[
L(x,\lambda) = 3x_1 + 7x_2 + 10x_3 + \lambda(7 - x_1 - 3x_2 - 5x_3) \\
= (3 - \lambda)x_1 + (10 - \lambda)x_2 + (5 - \lambda)x_3 + 7\lambda
\]
Dual objective: 

\[
\widehat{L}(\lambda) \equiv \min_{x_j \in \{0, 1\}, j=1,2,3} L(x, \lambda)
\]

\[
\widehat{L}(\lambda) = \min_{x_j \in \{0, 1\}} (3 - \lambda)x_1 + (10 - 3\lambda)x_2 + (5 - 5\lambda)x_3 + 7\lambda
\]

Given a value of \( \lambda \), the optimal \( x_j^*(\lambda) \) is 0 if its coefficient is positive, and 1 otherwise. For example, if \( \lambda = 2.5 \),

\[
L(x, 2.5) = 0.5x_1 - 0.5x_2 - 2.5x_3 + 17.5
\]

\[
x_1^*(2.5) = x_2^*(2.5) = 0, \ x_3^*(2.5) = 1
\]

\[
\widehat{L}(2.5) = 14.5
\]
Thus,

\[
x_1^*(\lambda) = \begin{cases} 
1 & \text{if } 3 - \lambda \leq 0, \quad \text{i.e., } \lambda \geq 3 \\
0 & \text{otherwise}
\end{cases}
\]

\[
x_2^*(\lambda) = \begin{cases} 
1 & \text{if } 7 - 3\lambda \leq 0, \quad \text{i.e., } \lambda \geq \frac{7}{3} \\
0 & \text{otherwise}
\end{cases}
\]

\[
x_3^*(\lambda) = \begin{cases} 
1 & \text{if } 10 - 5\lambda \leq 0, \quad \text{i.e., } \lambda \geq 2 \\
0 & \text{otherwise}
\end{cases}
\]

will minimize \(L(x, \lambda)\) for a given \(\lambda\)
<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$x_1^*(\lambda)$</th>
<th>$x_2^*(\lambda)$</th>
<th>$x_3^*(\lambda)$</th>
<th>$\tilde{L}(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq \lambda \leq 2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$7\lambda$</td>
</tr>
<tr>
<td>$2 \leq \lambda \leq \frac{7}{3}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$2\lambda + 10$</td>
</tr>
<tr>
<td>$\frac{7}{3} \leq \lambda \leq 3$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$-\lambda + 17$</td>
</tr>
<tr>
<td>$3 \leq \lambda \leq \infty$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$-2\lambda + 20$</td>
</tr>
</tbody>
</table>

When the coefficient of $x_j$ is zero, then both 0 & 1 are optimal values for that variable.
Dual problem:

Maximize $\widehat{L}(\lambda)$

s.t. $\lambda \geq 0$
By inspection of the graph of \( \widehat{L}(\lambda) \), we see that the optimal dual solution is

\[
\lambda^* = \frac{7}{3}, \quad \widehat{L}(\lambda^*) = \frac{44}{3}
\]

At \( \lambda^* \), both \( x' = (0,0,1) \) and \( x'' = (0,1,1) \) minimize \( L(x, \lambda) \).

But \( x' \) is infeasible in \( x_1 + 3x_2 + 5x_3 \geq 7 \) and \( x'' \) violates the complementary slackness condition:

\[
\lambda^* \left[ 7 - x'' - 3x'' - 5x'' \right] \neq 0
\]

Neither \( x' \) nor \( x'' \) are optimal in the primal problem.
Solving the primal problem by complete enumeration:

<table>
<thead>
<tr>
<th>$\mathbf{x}$</th>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$g(\mathbf{x})$</th>
<th>$f(\mathbf{x})$</th>
</tr>
</thead>
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<tr>
<td>0 0 0</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>0 0 1</td>
<td>2</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>0 1 0</td>
<td>4</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>0 1 1</td>
<td>-1</td>
<td>17</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>1 0 0</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1 0 1</td>
<td>1</td>
<td>13</td>
<td>13</td>
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</tr>
<tr>
<td>1 1 0</td>
<td>3</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>1 1 1</td>
<td>-2</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

infeasible

optimal in primal

infeasible

feasible
Primal solution
\[ \overline{L}(x^\prime) = 17 = 51/3 \]

Dual solution
\[ \widehat{L}(\lambda^*) = 44/3 \]

Duality Gap > 0!
\[ \overline{L}(x^\prime) - \widehat{L}(\lambda^*) = 7/3 \]
Graphical interpretation of the Duality Gap

primal optimum

\{ duality gap \}

dual optimum

G consists of the eight discrete points!
Consider the problem:

Minimize $f(x)$
subject to
$g_i(x) \leq 0$, $i = 1, 2, \ldots, m$
$x \in X$

where $f(x)$ & $g_i(x)$ are convex functions, and
$X$ is a convex set.

Let $\lambda \geq 0$ and $\bar{x} \in X$...
Saddlepoint Sufficiency Condition

Then \((\bar{x}, \bar{\lambda})\) is a saddlepoint of the Lagrangian function \(L(x, \lambda)\) if & only if

\[
\begin{cases}
\bullet & \bar{x} \text{ minimizes } L(x, \bar{\lambda}) = f(x) + \bar{\lambda}^T g(x) \text{ over } X \\
\bullet & g_i(\bar{x}) \leq 0 \text{ for each } i = 1, 2, \ldots, m \\
\bullet & \bar{\lambda}_i g_i(\bar{x}) = 0 \quad \text{which implies } f(\bar{x}) = L(\bar{x}, \bar{\lambda})
\end{cases}
\]

(If a saddlepoint exists, then the duality gap is zero!)

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If \((\overline{x}, \overline{\lambda})\) is a saddlepoint for \(L(x, \lambda)\)

then \(\overline{x}\) solves the primal problem:

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0, \ i = 1, 2, \ldots m \\
& \quad x \in X
\end{align*}
\]

and \(\overline{\lambda}\) solves the dual problem:

\[
\begin{align*}
\text{Maximize} & \quad \widehat{L}(\lambda) \\
\text{subject to} & \quad \lambda \geq 0
\end{align*}
\]

where \(\widehat{L}(\lambda) \equiv \min_{x \in X} L(x, \lambda)\)

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Consider the primal problem: Find

\[ \Phi = \inf \ f(x) \]

subject to \( g_i(x) \leq 0, \ i = 1, 2, \ldots m_1 \)

\( h_i(x) = 0, \ i = 1, 2, \ldots m_2 \)

\( x \in X \)

where \( X \subseteq \mathbb{R}^n \) is nonempty & convex

\( f(x) \& g_i(x) \) are convex

\( h_i(x) \) are linear

("infimum" may be replaced by "minimum" if the minimum is achieved at some \( x \).)
Define the Dual Problem:

Find

\[ \Psi = \sup_{\lambda \geq 0} \, \tilde{L}(\lambda, \mu) \]

where

\[ \tilde{L}(\lambda, \mu) \equiv \inf_{x \in X} \{ f(x) + \lambda^T g(x) + \mu^T h(x) \} \]
Assume also that the following "Constraint Qualification" holds:

There exists $\tilde{x}$ such that

$$g_i(\tilde{x}) < 0, \ i = 1, 2, \ldots, m_1$$

$$h_i(\tilde{x}) = 0, \ i = 1, 2, \ldots, m_2$$

$$& 0 \in \text{int} \ h(X)$$
Then \( \Phi = \Psi \)

\( i.e., \) there is no duality gap!

Furthermore, if \( \Phi > -\infty \) then

- \( \Psi = \widehat{\lambda}(\lambda^*, \mu^*) \) for some \( \lambda^* \geq 0 \)
- if \( x^* \) solves the primal, it satisfies complementary slackness, i.e.,
  \[ \lambda_i^* g_i(x^*) = 0 \ \forall \ i \]
EXAMPLE

Minimize $f(x) = -x^2 - x^3$
subject to $x^2 \leq 1$

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?
Graphically, we can see that \( x^* = 1, \ f(x^*) = -2 \)

\[
f(x) = -x^2 - x^3
\]
Lagrangian function

\[ L(x, \lambda) = -x^2 - x^3 + \lambda (x^2 - 1) \]

KKT Conditions

\[ \frac{dL}{dx} = -2x - 3x^2 + 2\lambda \quad x = 0 \]
\[ x^2 \leq 1 \]
\[ \lambda (x^2 - 1) = 0 \]
\[ \lambda \geq 0 \]

KKT points are
\[ (x, \lambda) = \left(-\frac{2}{3}, 0\right), (0, 0), (1, \frac{5}{2}) \]
\[ L(x, \lambda) = -\frac{4}{27}, 0, -2 \]
Dual Problem

Maximize $\widetilde{L}(\lambda)$
subject to $\lambda \geq 0$

where $\widetilde{L}(\lambda) \equiv \min_{x \in X} L(x, \lambda)$

\[
= \min_{x \in X} \left\{ -x^2 - x^3 + \lambda(x^2 - 1) \right\}
\]

\[
= -\infty \text{ for all } \lambda \geq 0
\]

$\implies$ Maximum $\widetilde{L}(\lambda) = -\infty$
\[ G = \{ (z_1, z_2) \mid z_1 = g(x), z_2 = f(x) \text{ for some } x \} \]

\[
\begin{aligned}
\begin{cases}
  z_2 = f(x) = -x^2 - x^3 \\
  z_1 = g(x) = x^2 - 1 \implies x = \pm (1+z_1)^{1/2}
\end{cases}
\end{aligned}
\]

\[ \implies G = \{ (z_1, z_2) \mid z_2 = -(1+z_1) \pm (1+z_1)^{3/2} \} \]
The set $G$ consists of the curve below:

There is no nonvertical support of $G$ which has negative ($=-\lambda$) slope!
EXAMPLE

Minimize $-(x - 4)^2$

subject to $1 \leq x \leq 6$

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?
EXAMPLE

Minimize \( f(x,y) = x \)
subject to
\[ g(x,y) = x^2 + y^2 \leq 1 \]

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?
EXAMPLE

Minimize \((x - 4)^2\)
subject to
\[1 \leq x \leq 3\]

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?

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