DUALITY IN LINEAR PROGRAMMING

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The SYMMETRIC primal/dual pair:

Primal:

Minimize $c^t x$

subject to:

$$Ax \geq b$$
$$x \geq 0$$

Dual:

Maximize $b^t y$

subject to:

$$A^t y \leq c$$
$$y \geq 0$$

where A is an mxn matrix, x&c are vectors of length n, and y&b are vectors of length m. (Note: A^t denotes transpose of the matrix A.)

Note the following characteristics:

- the primal LP is mxn, i.e., m constraints (not including nonnegativity) and n variables
- the dual LP is nxm, i.e., n constraints (not including nonnegativity) and m variables

Primal:	*Dual:*
minimize $c^t x$	Maximize $b^t y$
subject to:	subject to:
$Ax \geq b$	$A^t y \leq c$
$x \geq 0$	$y \geq 0$

Note the following characteristics:

- for every variable in the primal problem, there is a corresponding inequality constraint in the dual problem
- for every inequality constraint (not including nonnegativity), there is a corresponding dual variable

Primal:

minimize $c^t x$

subject to:

$$Ax \geq b$$
$$x \geq 0$$

Dual:

Maximize $b^t y$

subject to:

$$A^t y \leq c$$
$$y \geq 0$$

Note the following characteristics:

- the right-hand-side vector (b) of the primal problem serves as the objective function coefficient vector of the dual problem.

Primal:

minimize $c^t x$

subject to:

$$Ax \geq b$$
$$x \geq 0$$

Dual:

Maximize $b^t y$

subject to:

$$A^t y \leq c$$
$$y \geq 0$$

Example

Primal

Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \geq 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Dual

Maximize $6y_1 + 8y_2$

subject to:

$$5y_1 + 2y_2 \leq 20$$

$$y_1 + 2y_2 \leq 10$$

$$y_1 \geq 0,\ y_2 \geq 0$$

Primal

Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \geq 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Dual

Maximize $6y_1 + 8y_2$

subject to:

$$5y_1 + 2y_2 \leq 20$$

$$y_1 + 2y_2 \leq 10$$

$$y_1 \geq 0, \quad y_2 \geq 0$$

The primal problem is a MINIMIZATION with ≥ constraints, while the dual problem is a MAXIMIZATION with ≤ constraints!

Primal	*Dual*
Minimize $20x_1 + 10x_2$	Maximize $6y_1 + 8y_2$
subject to:	subject to:
$5x_1 + x_2 \geq 6$	$5y_1 + 2y_2 \leq 20$
$2x_1 + 2x_2 \geq 8$	$y_1 + 2y_2 \leq 10$
$x_1 \geq 0, \; x_2 \geq 0$	$y_1 \geq 0, \; y_2 \geq 0$

The objective coefficients of the primal serve as the right-hand-side of the dual problem!

Primal

Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \geq 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Dual

Maximize $6y_1 + 8y_2$

subject to:

$$5y_1 + 2y_2 \leq 20$$

$$y_1 + 2y_2 \leq 10$$

$$y_1 \geq 0,\ y_2 \geq 0$$

... and conversely, the right-hand-side of the primal problem serves as objective coefficients of the dual problem!

Primal	*Dual*
Minimize $20x_1 + 10x_2$	Maximize $6y_1 + 8y_2$
subject to:	subject to:
$5x_1 + x_2 \geq 6$	$5y_1 + 2y_2 \leq 20$
$2x_1 + 2x_2 \geq 8$	$y_1 + 2y_2 \leq 10$
$x_1 \geq 0,\ x_2 \geq 0$	$y_1 \geq 0,\ y_2 \geq 0$

To every constraint in the primal, there corresponds a dual variable.

Primal	*Dual*
Minimize $20x_1 + 10x_2$	Maximize $6y_1 + 8y_2$
subject to:	subject to:
$5x_1 + x_2 \geq 6$	$5y_1 + 2y_2 \leq 20$
$2x_1 + 2x_2 \geq 8$	$y_1 + 2y_2 \leq 10$
$x_1 \geq 0,\ x_2 \geq 0$	$y_1 \geq 0,\ y_2 \geq 0$

To every variable in the primal problem, there corresponds a constraint in the dual problem.

Primal

Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \geq 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Dual

Maximize $6y_1 + 8y_2$

subject to:

$$5y_1 + 2y_2 \leq 20$$

$$y_1 + 2y_2 \leq 10$$

$$y_1 \geq 0, \; y_2 \geq 0$$

Both primal and dual problems include nonnegativity constraints on the variables.

Suppose that we have an inequality reversed in the primal problem, for example:

Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \le 6$$

note the reversed direction!

$$2x_1 + 2x_2 \ge 8$$

$$x_1 \ge 0,\ x_2 \ge 0$$

How do we write the dual of this problem?

First we must transform the problem:

We multiply the offending inequality by −1, thereby reversing the direction of the inequality:

Minimize $20x_1 + 10x_2$	Minimize $20x_1 + 10x_2$
subject to:	subject to:
$5x_1 + x_2 \le 6$ ⟹	$-5x_1 - x_2 \ge -6$
$2x_1 + 2x_2 \ge 8$	$2x_1 + 2x_2 \ge 8$
$x_1 \ge 0,\ x_2 \ge 0$	$x_1 \ge 0,\ x_2 \ge 0$

Now the problem is in the form of the primal in the symmetric primal/dual pair. We can therefore write its DUAL problem:

Minimize $20x_1 + 10x_2$

subject to:

$$-5x_1 - x_2 \geq -6$$
$$2x_1 + 2x_2 \geq 8$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Maximize $-6y_1 + 8y_2$

subject to

$$-5y_1 + 2y_2 \leq 20$$
$$-y_1 + 2y_2 \leq 10$$
$$y_1 \geq 0,\ y_2 \geq 0$$

It is interesting to now make a change of variable: let $y_1' = -y_1$

Maximize $-6y_1 + 8y_2$

subject to

$$-5y_1 + 2y_2 \le 20$$
$$-y_1 + 2y_2 \le 10$$
$$y_1 \ge 0, \quad y_2 \ge 0$$

$\Longrightarrow$

Maximize $6y_1' + 8y_2$

subject to

$$5y_1' + 2y_2 \le 20$$
$$y_1' + 2y_2 \le 10$$
$$y_1' \le 0, \quad y_2 \ge 0$$

↑

Same as dual of the symmetric primal/dual pair, except for non-positivity replacing non-negativity!

Suppose that, rather than an inequality constraint, we had an equality constraint. For example:

Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 = 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0, \; x_2 \geq 0$$

What is its DUAL problem?

We must first transform the equality constraint into equivalent inequalities:

$$5x_1 + x_2 = 6 \Longrightarrow \begin{cases} 5x_1 + x_2 \geq 6 \\ 5x_1 + x_2 \leq 6 \end{cases} \Longrightarrow \begin{cases} 5x_1 + x_2 \geq 6 \\ -5x_1 - x_2 \geq -6 \end{cases}$$

So our problem, in the form of the primal in the symmetric primal/dual pair, is:

$$\begin{aligned} \text{Minimize } & 20x_1 + 10x_2 \\ \text{s.t. } & 5x_1 + x_2 \geq 6 \\ & -5x_1 - x_2 \geq -6 \\ & 2x_1 + 2x_2 \geq 8 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

We can now write its DUAL problem:

(For reasons to be apparent, we choose to name our dual variables not y_1, y_2, *and* y_3 *but* y'_1, y''_1, *and* y_2 *.)*

P: Minimize $20x_1 + 10x_2$

s.t.

$$5x_1 + x_2 \geq 6$$
$$-5x_1 - x_2 \geq -6$$
$$2x_1 + 2x_2 \geq 8$$
$$x_1 \geq 0,\ x_2 \geq 0$$

D: Max $6y'_1 - 6y''_1 + 8y_2$

s.t.

$$5y'_1 - 5y''_1 + 2y_2 \leq 20$$
$$y'_1 - y''_1 + 2y_2 \leq 10$$
$$y'_1 \geq 0,\ y''_1 \geq 0,\ y_2 \geq 0$$

Notice that the pair of dual variables y_1' and y_1'' always appear with opposite signs:

$$\begin{array}{ll} \text{Max } 6y_1' - 6y_1'' + 8y_2 \\ \text{s.t.} \\ \quad 5y_1' - 5y_1'' + 2y_2 \le 20 \\ \quad y_1' - y_1'' + 2y_2 \le 10 \\ \quad y_1' \ge 0,\ y_1'' \ge 0,\ y_2 \ge 0 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \text{Max } 6(y_1' - y_1'') + 8y_2 \\ \text{s.t.} \\ \quad 5(y_1' - y_1'') + 2y_2 \le 20 \\ \quad (y_1' - y_1'') + 2y_2 \le 10 \\ \quad y_1' \ge 0,\ y_1'' \ge 0,\ y_2 \ge 0 \end{array}$$

It is instructive now to make the change of variable: $y_1 = y_1' - y_1''$

Letting $y_1 = y_1' - y_1''$,

$$\begin{aligned} &\text{Max } 6(y_1' - y_1'') + 8y_2 \\ &\text{s.t.} \\ &\quad 5(y_1' - y_1'') + 2y_2 \le 20 \\ &\quad (y_1' - y_1'') + 2y_2 \le 10 \\ &\quad y_1' \ge 0,\ y_1'' \ge 0,\ y_2 \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\text{Maximize } 6y_1 + 8y_2 \\ &\text{subject to} \\ &\quad 5y_1 + 2y_2 \le 20 \\ &\quad y_1 + 2y_2 \le 10 \\ &\quad y_2 \ge 0 \end{aligned}$$

(We cannot include a constraint on the sign of y_1*, since it is the difference of two variables.)*

This is the same as the dual in the symmetric primal/dual pair, except for the missing nonnegativity restriction!

We next show:

The dual of the DUAL problem

is the PRIMAL problem!

Problem (P):

Minimize $c^t x$

subject to:

$$Ax \geq b$$
$$x \geq 0$$

Problem (D):

Maximize $b^t y$

subject to:

$$A^t y \leq c$$
$$y \geq 0$$

How do we write the DUAL of problem (D) above? First we must write it as a minimization problem with ≥ constraints.

Problem (D):		Problem (D'):
Maximize $b^t y$	(equivalent) $\Longleftrightarrow$	$-$Min $(-b^t)y$
subject to:		subject to
$A^t y \le c$		$(-A^t)y \ge -c$
$y \ge 0$		$y \ge 0$

MINIMIZING the NEGATIVE of a function yields the same solution (except for sign) as MAXIMIZING the function.
NEGATING both sides of a ≤ constraint produces a ≥ constraint.

Primal:		*Dual:*
Minimize $c^t x$ subject to: $Ax \geq b$ $x \geq 0$	$\Longrightarrow$	Maximize $b^t y$ subject to: $A^t y \leq c$ $y \geq 0$
Problem (D'):		**Problem (DD'):**
$-$Min $(-b^t)y$ subject to $(-A^t)y \geq -c$ $y \geq 0$	$\Longrightarrow$	$-$Max $(-c^t)u$ subject to $(-A^t)^t u \leq (-b^t)^t$ $u \geq 0$

Problem (DD'):

$$-\text{Max } (-c^t)u \quad \text{subject to} \quad (-A^t)^t u \le (-b^t)^t, \quad u \ge 0$$

is equivalent to: $\Longrightarrow$

$$\text{Min } c^t u \quad \text{subject to} \quad Au \ge b, \quad u \ge 0$$

which is the same as the original PRIMAL problem (P), except for the name of the variables (u instead of x), which is arbitrary!

The dual of the DUAL problem is the PRIMAL problem!

So, given a primal/dual pair of LP problems, it is arbitrary which is referred to as the primal, and which is referred to as the dual.

Writing the dual of a general Primal LP

The dual of an LP may be found by first rewriting the LP in the form of one of the LPs in the symmetric Primal/Dual pair.

On the other hand, the dual can be written directly for any LP using the following relationships.

Maximize		Minimize
Type of constraint i	⟷	Sign of variable i
$\le$	⟷	nonnegative
$=$	⟷	unrestricted
$\ge$	⟷	nonpositive
Sign of variable j	⟷	Type of constraint j
nonnegative	⟷	$\ge$
unrestricted	⟷	$=$
nonpositive	⟷	$\le$

Example

We want to write the dual of the LP:

Minimize $6X_1 + 3X_2 + 5X_4$

s.t.

$$X_1 - 2X_2 + 4X_3 \leq 20$$

$$2X_1 + X_2 - X_4 = 30$$

$$5X_2 + X_3 + X_4 \geq 50$$

$X_1 \geq 0$, X_2 urs, $X_3 \geq 0$, $X_4 \leq 0$

We immediately notice that dual will be MAX and size of problem will be 4x3

Minimize $6X_1 + 3X_2 + 5X_4$

s.t.

$$X_1 - 2X_2 + 4X_3 \le 20$$

$$2X_1 + X_2 - X_4 = 30$$

$$5X_2 + X_3 + X_4 \ge 50$$

$X_1 \ge 0$, X_2 urs, $X_3 \ge 0$, $X_4 \le 0$

Dual will be a maximization

 # dual variables =3

#dual constraints =4

Primal	Dual
Minimize $6X_1 + 3X_2 + 5X_4$	Maximize ___ Y_1 + ___ Y_2 + ___ Y_3
s.t.	s.t.
$X_1 - 2X_2 + 4X_3 \le 20$	___ Y_1 + ___ Y_2 + ___ Y_3 ___ ___
$2X_1 + X_2 - X_4 = 30$	___ Y_1 + ___ Y_2 + ___ Y_3 ___ ___
$5X_2 + X_3 + X_4 \ge 50$	___ Y_1 + ___ Y_2 + ___ Y_3 ___ ___
	___ Y_1 + ___ Y_2 + ___ Y_3 ___ ___
$X_1 \ge 0$, X_2 urs, $X_3 \ge 0$, $X_4 \le 0$	Y_1 ___, Y_2 ___, Y_3 ___

Transposing coefficients and right-hand-sides:

Primal	Dual
Minimize $6X_1 + 3X_2 + 5X_4$	Maximize $20Y_1 + 30Y_2 + 50Y_3$
s.t. $X_1 - 2X_2 + 4X_3 \leq 20$ $2X_1 + X_2 - X_4 = 30$ $5X_2 + X_3 + X_4 \geq 50$	s.t. $1\ Y_1 + 2\ Y_2 + 0\ Y_3$ .... 6 $-2\ Y_1 + 1\ Y_2 + 5\ Y_3$ .... 3 $4\ Y_1 + 0\ Y_2 + 1\ Y_3$ .... 0 $0\ Y_1 - 1\ Y_2 + 1\ Y_3$ 5
$X_1 \geq 0, X_2$ urs$, X_3 \geq 0, X_4 \leq 0$	Y_1 ...., Y_2 ...., Y_3 ....

Determining sign restrictions of dual variables

Primal	Dual
Minimize $6X_1+3X_2+X_4$	Maximize $20Y_1+30Y_2+50Y_3$
s.t.	s.t.
$X_1-2X_2+4X_3 \le 20$	$1Y_1+2Y_2+0Y_3$ 6
$2X_1+X_2-X_4=30$	$-2Y_1+1Y_2+5Y_3$ 3
$5X_2+X_3+X_4 \ge 50$	$4Y_1+0Y_2+1Y_3$ 0
	$0Y_1-1Y_2+1Y_3$ 5
$X_1 \ge 0, X_2$ urs, $X_3 \ge 0, X_4 \le 0$	$Y_1 \le 0$, Y_2 **urs**, $Y_3 \ge 0$

Min	Max
$\ge$	nonnegative
$=$	urs
$\le$	nonpositive

Determining form of dual constraints:

Primal	Dual
Minimize $6X_1 + 3X_2 + X_4$	Maximize $20Y_1 + 30Y_2 + 50Y_3$
s.t.	s.t.
$X_1 - 2X_2 + 4X_3 \le 20$	$1\,Y_1 + 2\,Y_2 + 0\,Y_3 \le 6$
$2X_1 + X_2 - X_4 = 30$	$-2\,Y_1 + 1\,Y_2 + 5\,Y_3 = 3$
$5X_2 + X_3 + X_4 \ge 50$	$4\,Y_1 + 0\,Y_2 + 1\,Y_3 \le 0$
	$0\,Y_1 - 1\,Y_2 + 1\,Y_3 \ge 5$
$X_1 \ge 0, X_2$ urs, $X_3 \ge 0, X_4 \le 0$	$Y_1 \le 0, Y_2$ urs , $Y_3 \ge 0$

Min	Max
nonnegative	$\le$
urs	$=$
nonpositive	$\ge$

The WEAK Duality Theorem:

Consider the symmetric primal/dual pair:

Primal:

Minimize $c^t x$
subject to:
$$Ax \ge b$$
$$x \ge 0$$

Dual:

Maximize $b^t y$
subject to:
$$A^t y \le c$$
$$y \ge 0$$

Suppose that $\hat{x}$ is feasible in the primal problem, and $\hat{y}$ is feasible in the dual. Then $c^t\hat{x} \ge b^t\hat{y}$

Proof

The proof of the Weak Duality Theorem is very simple:

$$A\hat{x} \geq b \;\; \& \;\; \hat{y} \geq 0 \quad \Longrightarrow \quad \hat{y}^t A\hat{x} \geq \hat{y}^t b$$

Transpose $A^t\hat{y} \leq c$ to get $(A^t\hat{y})^t \leq c^t$, i.e. $\hat{y}^t A \leq c^t$

Then $\hat{y}^t A \leq c^t \;\; \& \;\; \hat{x} \geq 0 \Longrightarrow \hat{y}^t A\hat{x} \leq c^t\hat{x}$

Combining these two inequalities gives us

$$c^t\hat{x} \geq \hat{y}^t A\hat{x} \geq \hat{y}^t b \quad \Longrightarrow \quad c^t\hat{x} \;\geq\; b^t\hat{y}$$

Corollaries of the Weak Duality Theorem:

If x^* and y^* are optimal solutions of the primal and dual problems, respectively:

- objective value for any primal feasible solution is greater than or equal to $b^t y^*$
- objective value for any dual feasible solution is less than or equal $c^t x^*$

Corollaries of the Weak Duality Theorem (continued):

- if $\hat{x}$ and $\hat{y}$ are feasible in the primal & dual problems, respectively, and if $c^t\hat{x} = b^t\hat{y}$, then $\hat{x}$ = primal optimum (x^*)
  $\hat{y}$ = dual optimum (y^*)
- if the primal is feasible and unbounded below, then the dual problem must be infeasible!
- if the dual is feasible and unbounded above, then the primal problem must be infeasible!

Theorem:

If B^* is an optimal basis of the primal problem (P) then the simplex multiplier vector π^* relative to the basis B^* is an optimal solution to the dual problem (D).

(The simplex multiplier vector π may be computed by the formula

$$\pi^* = c_B^t (A^B)^{-1} \quad .)$$

Proof

Proof:

Let's write the problem (P) with equality constraints, as required by the simplex method:

Primal:

Minimize $c^t x$
subject to:

$$Ax \geq b$$
$$x \geq 0$$

$\Longrightarrow$

Minimize $[c \,\vdots\, 0] \begin{bmatrix} x \\ s \end{bmatrix}$ ← *surplus variable*

subject to

$$[A \,\vdots\, -\mathbf{I}] \begin{bmatrix} x \\ s \end{bmatrix} = b$$

$$x \geq 0$$

identity matrix → $\mathbf{I}$

Suppose π^* is the optimal simplex multiplier vector. Then the optimality conditions (for terminating the simplex algorithm) must be satisfied, namely

$$\begin{pmatrix}\textit{cost of} \\ \textit{variable}\end{pmatrix} - \pi^* \begin{pmatrix}\textit{column of} \\ \textit{constraint} \\ \textit{coefficients}\end{pmatrix} \geq 0$$

$$c_j \quad - \quad \text{"}z_j\text{"}$$

"reduced cost"

These conditions must be satisfied for both the original variables (x) and the surplus variables (s):

$$\begin{pmatrix}\textit{cost of} \\ \textit{variable}\end{pmatrix} - \pi^* \begin{pmatrix}\textit{column of} \\ \textit{constraint} \\ \textit{coefficients}\end{pmatrix} \geq 0$$

Minimize $\left[c \,\vdots\, 0\right]\begin{bmatrix} x \\ \hdashline s \end{bmatrix}$

subject to

$$\left[A \,\vdots\, -\mathbf{I}\right]\begin{bmatrix} x \\ \hdashline s \end{bmatrix} = b$$

$c^t - \pi^* A \geq 0$, i.e. $c^t \geq \pi^* A$

$0 - \pi^* (-\mathbf{I}) \geq 0$, i.e. $\underbrace{\pi^* \geq 0}$

feasibility conditions for the dual!

And so if π^* is the optimal simplex multiplier,

$$\pi^* A \leq c^t \text{, i.e. } A^t \pi^* \leq c$$
$$\pi^* \geq 0$$

i.e., π^* is feasible in the dual problem.

Recall the computation of π^*: $\pi^* = c_B^t (A^B)^{-1}$

Recall also that $x_B^* = (A^B)^{-1} b$

Therefore $c^t x^* = c_B^t x_B^* = \underbrace{c_B^t (A^B)^{-1}}_{\pi^*} b = \pi^* b$

↑ *since nonbasic variables are zero!*

Therefore, π^* is feasible in the dual, and the objective functions of the primal & dual problems evaluated at x^* and π^*, respectively, are equal.

Hence, by a corollary of the WEAK DUALITY THEOREM, x^* and π^* must both be optimal in their respective problems!

Q.E.D.

Example:

P: Maximize $4X_1 + 5X_2$
subject to

$$X_1 + X_2 \le 8$$
$$3X_1 + 2X_2 \le 18$$
$$2X_1 + 5X_2 \le 15$$
$$5X_1 - X_2 \le 10$$
$$X_1 \ge 0, X_2 \ge 0$$

This problem has 2 variables & 4 inequality constraints, and so its dual will have 4 variables and 2 inequality constraints.

The dual problem:

D: Minimize $8Y_1 + 18Y_2 + 15Y_3 + 10Y_4$

subject to:

$$Y_1 + 3Y_2 + 2Y_3 + 5Y_4 \geq 4$$

$$Y_1 + 2Y_2 + 5Y_3 - Y_4 \geq 5$$

$$Y_1 \geq 0,\ Y_2 \geq 0,\ Y_3 \geq 0,\ Y_4 \geq 0$$

This dual problem has fewer constraints than its primal, and, when solved by the simplex method, usually requires

- ***fewer iterations*** *(typically 1.5m to 2m iterations)*
- ***fewer computations per iteration*** *(especially if using the revised simplex!)*

The optimal simplex tableau for the dual problem is

basic	$-z$	Y_1	Y_2	Y_3	Y_4	S_1	S_2	RHS	
$-z$	1	$\frac{96}{27}$	$\frac{181}{27}$	0	0	$\frac{65}{27}$	$\frac{55}{27}$	$-\frac{505}{27}$	← *reduced cost row*
Y_4	0	$\frac{3}{27}$	$\frac{11}{27}$	0	1	$-\frac{5}{27}$	$\frac{2}{27}$	$\frac{14}{27}$	
Y_3	0	$\frac{6}{27}$	$\frac{13}{27}$	1	0	$-\frac{1}{27}$	$-\frac{5}{27}$	$\frac{19}{27}$	

surplus variables (S_1, S_2)

The optimal solution to the dual is

$$Y_1 = Y_2 = 0,\ Y_3 = \frac{19}{27},\ Y_4 = \frac{14}{27}$$

What is the optimal solution of the primal problem?

The Simplex Multiplier vector for the optimal dual tableau is

$$\pi = \left[\frac{65}{27} , \ \frac{55}{27} \right]$$

(Why? the reduced cost of the surplus variable S_1 is its cost minus π times the column of coefficients:

reduced cost of S_1 *is* $0 - \pi \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \pi_1$

Likewise, the reduced cost of the surplus variable for row # i is the Simplex Multiplier for that row.)

Therefore, the optimal solution of the original primal problem is

$$X_1 = \frac{65}{27} \qquad X_2 = \frac{55}{27}$$

Thus, we may choose to solve either the primal or the dual problem, whichever is easier, and obtain the solution to both simultaneously!

Note: $-\pi_i$ appears as the reduced cost of a slack variable in row i.

If there is a surplus variable in row i, its reduced cost is $0-(-1)\pi_i = +\pi_i$.

If constraint i is an equation without slack or surplus variable, then π_i will NOT appear in the optimal tableau!

The Fundamental Duality Theorem:

Problem (P):

Minimize $c^t x$
subject to:

$$Ax \geq b$$
$$x \geq 0$$

Problem (D):

Maximize $b^t y$
subject to:

$$A^t y \leq c$$
$$y \geq 0$$

- If both problems (P) & (D) are feasible, then both have an optimal solution and their optimal values are equal, i.e.,

$$c^t x^* = b^t y^*$$

- If one of the problems [either (P) or (D)] has an unbounded objective, then the other problem is infeasible.
- If only one of the problems is feasible, then its objective must be unbounded over the feasible region.

Proof

Note that it is possible that BOTH primal and dual problems are infeasible.

Example:

Primal

Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \geq 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Dual

Maximize $6y_1 + 8y_2$

subject to:

$$5y_1 + 2y_2 \leq 20$$

$$y_1 + 2y_2 \leq 10$$

$$y_1 \geq 0, \; y_2 \geq 0$$

The primal system has 6 basic solutions, of which 3 are feasible:

Primal

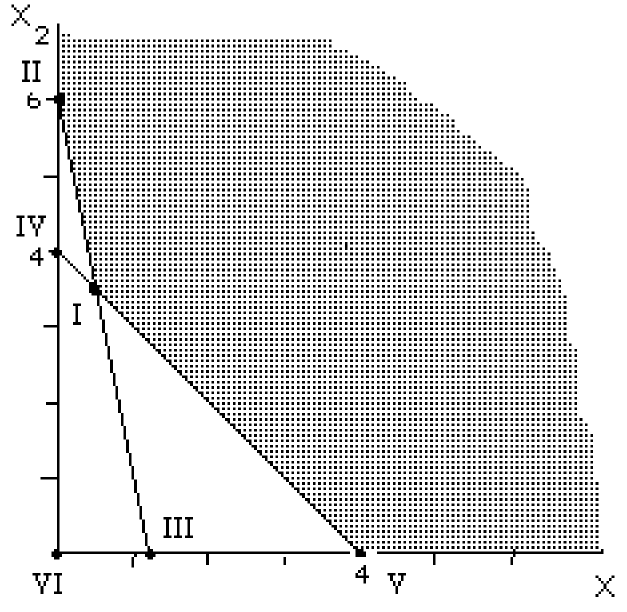
Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \geq 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0, \; x_2 \geq 0$$



The dual system also has six basic solutions, 4 of them feasible:

Dual

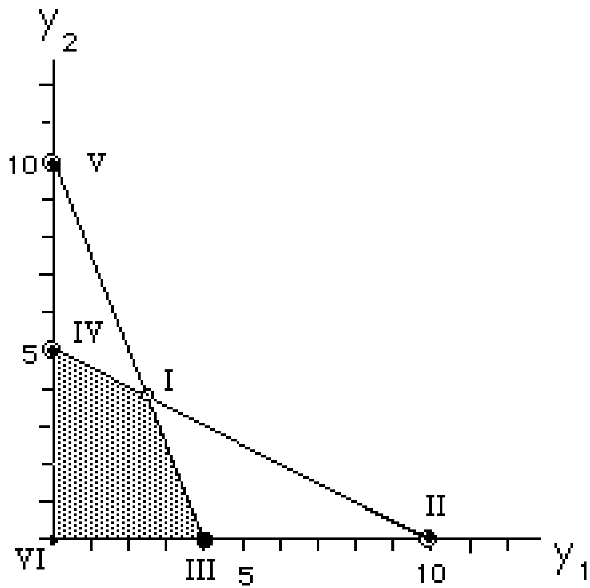
Maximize $6y_1 + 8y_2$

subject to:

$$5y_1 + 2y_2 \le 20$$

$$y_1 + 2y_2 \le 10$$

$$y_1 \ge 0, \ y_2 \ge 0$$



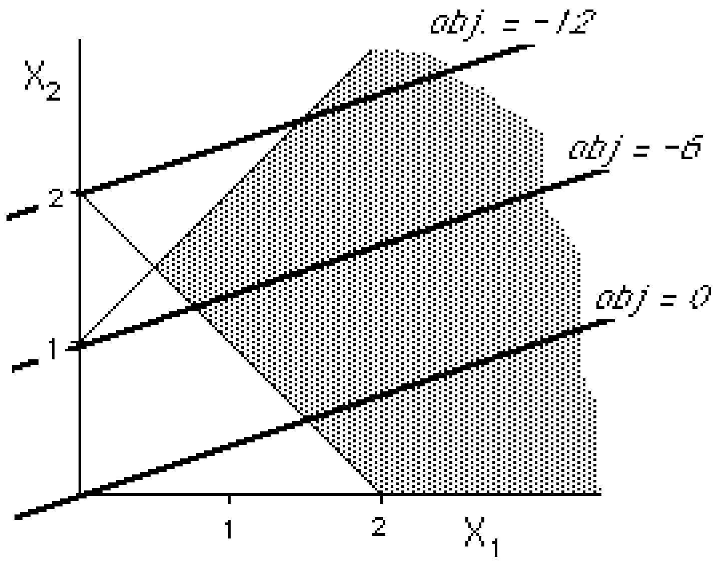
	PRIMAL							DUAL				
extreme pt. #	X_1	X_2	X_3	X_4	feasible?	obj.	feasible?	Y_1	Y_2	Y_3	Y_4	
VI	4	0	14	0	√	80		0	10	0	-10	
II	0	6	0	4	√	60		10	0	-30	0	
I	.5	3.5	0	0	√	45	√	2.5	3.75	0	0	⟵ optimal
IV	0	4	-2	0		40	√	0	5	10	0	
III	1.2	0	0	-5		24	√	4	0	0	6	6
VI	0	0	-6	-8		0	√	0	0	20	10	

Example: Unbounded Primal Problem

Minimize $2X_1 - 6X_2$
subject to

$$X_1 + X_2 \geq 2$$
$$X_1 - X_2 \geq -1$$
$$X_1 \geq 0, \quad X_2 \geq 0$$

The objective --> - ∞ if we travel along the edge of the feasible region to the upper right!



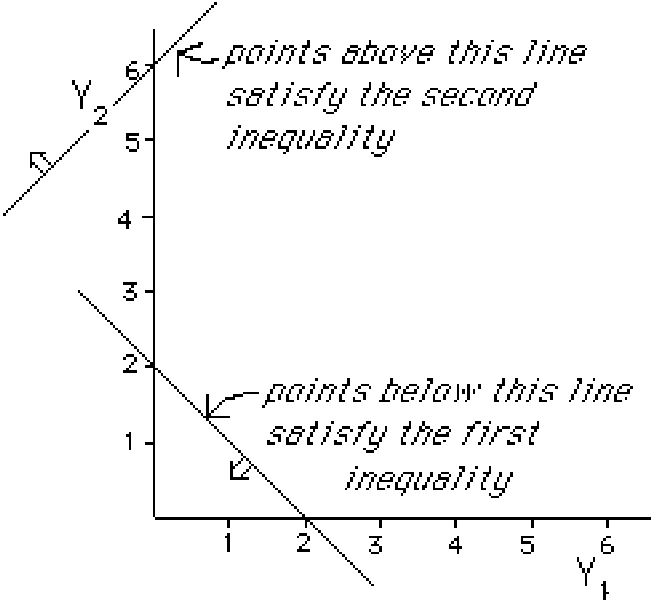
The dual of this unbounded primal problem:

Maximize $2Y_1 - Y_2$
subject to

$$Y_1 + Y_2 \le 2$$
$$Y_1 - Y_2 \le -6$$

$$Y_1 \ge 0,\ Y_2 \ge 0$$

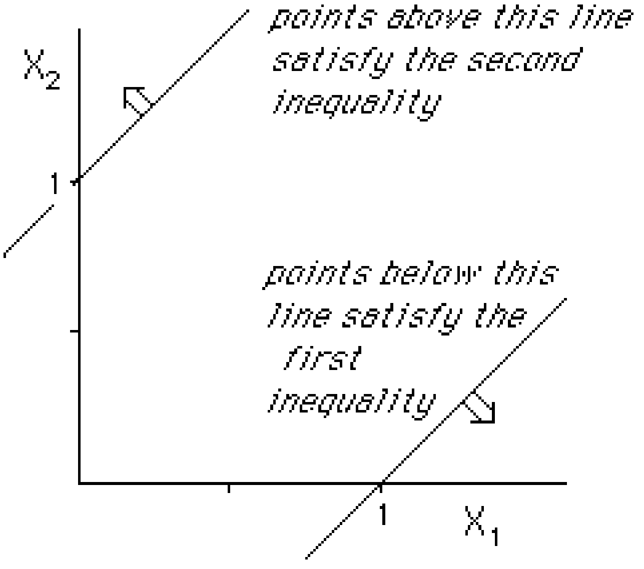
(Infeasible!)



Example: Infeasible Primal Problem

Minimize $X_1 + X_2$
subject to

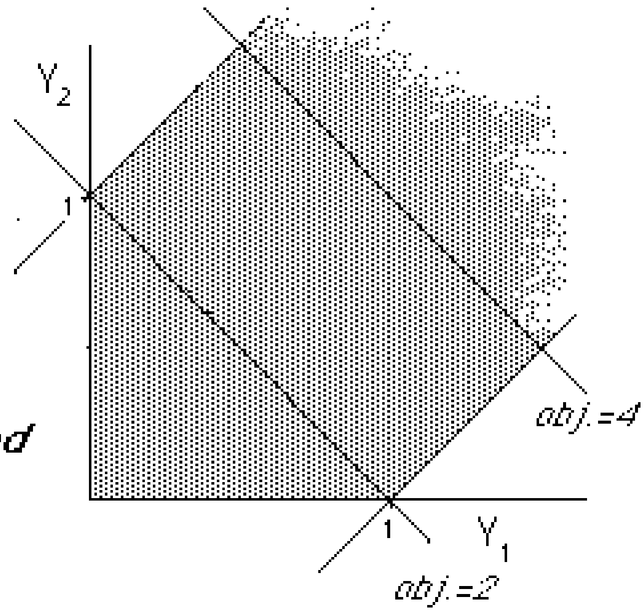
$$X_1 - X_2 \geq 1$$
$$-X_1 + X_2 \geq 1$$
$$X_1 \geq 0,\ X_2 \geq 0$$



Dual of the infeasible primal problem:

Maximize $Y_1 + Y_2$
subject to

$$Y_1 - Y_2 \le 1$$
$$-Y_1 + Y_2 \le 1$$
$$Y_1 \ge 0, \; Y_2 \ge 0$$



Objective is unbounded as we move to the upper right!

Recall that it is possible that BOTH primal and dual problems are infeasible!

Can you alter the preceding (infeasible) primal problem so that the dual problem becomes infeasible (while the primal problem remains infeasible)?

Hint: Leave the primal constraints unchanged. Can you then change the dual RHS (=primal objective coefficients) so that the dual becomes infeasible?

An economic interpretation of the LP dual problem:

Consider the DIET PROBLEM:

A housewife has to find a minimum-cost diet for her family by selecting from among 5 foods, subject to the constraints that the diet will provide at least 21 units of vitamin A and 12 units of vitamin B per person per day:

Food:	1	2	3	4	5	
Vit. A content	1	0	1	1	2	units/oz.
Vit. B content	0	1	2	1	1	units/oz.
Cost	20	20	31	11	12	¢/oz.

The housewife's LP model:

Minimize $20x_1 + 20x_2 + 31x_3 + 11x_4 + 12x_5$ ← *cost per person per day*

subject to

$$x_1 + x_3 + x_4 + 2x_5 \geq 21 \quad \leftarrow \text{Vit. A rqmt.}$$

$$x_2 + 2x_3 + x_4 + x_5 \geq 12 \quad \leftarrow \text{Vit. B rqmt.}$$

$$x_1, \ldots x_5 \geq 0$$

where x_j = quantity of food #j (oz./day) per person

(She is ignoring requirements for all other nutrients, and consideration of pallatability, etc.)

The Pill Salesman's Problem:

Consider a door-to-door salesman of vitamin pills. He has a supply of vitamin A pills (1 unit each) and vitamin B pills (also 1 unit each).

He visits the housewife and suggests that she buy pills from him to feed her family, rather than the foods #1 through #5.

In order to be competitive with the grocery, she must be able to feed her family pills for a cost no more than that of her least-cost meal. *(We ignore the value of her labor!)*

The Pill Salesman's LP problem:

Choose prices of the pills:

π_A = price per unit of vitamin A pill

π_B = price per unit of vitamin B pill

so as to Maximize $21\pi_A + 12\pi_B$ ← *revenue (¢/day/person)*

subject to

$$\begin{aligned} \pi_A &\leq 20 \\ \pi_B &\leq 20 \\ \pi_A + 2\pi_B &\leq 31 \\ \pi_A + \pi_B &\leq 11 \\ 2\pi_A + \pi_B &\leq 12 \end{aligned}$$

the pill-equivalent of each food must cost no more than the food itself (applies to the five constraints above)

$\pi_A \geq 0, \ \pi_B \geq 0$

But this is the DUAL of the housewife's LP problem:

$$\text{Minimize } 20x_1 + 20x_2 + 31x_3 + 11x_4 + 12x_5$$

subject to

$$\begin{aligned} x_1 \quad\quad\quad + x_3 + x_4 + 2x_5 &\geq 21 \\ x_2 + 2x_3 + x_4 + x_5 &\geq 12 \\ x_1, \ldots x_5 &\geq 0 \end{aligned}$$

The Fundamental Duality Theorem tells us that (if both problems are feasible & bounded) the two LP problems have the same optimal values!

That is, the housewife would be indifferent between preparing the meals & serving the pills.

The FARKAS Lemma:

The following statements are equivalent:

(i) if $y^t A \leq 0$ for some y, then $y^t b \leq 0$

&

(ii) the system $Ax=b, x \geq 0$ is feasible

(This "Lemma" is of great theoretical importance in optimization, and is used in the proof of the Kuhn-Tucker optimality conditions in nonlinear programming.)

Proof

Proof of the FARKAS Lemma:

Consider the following primal/dual pair of LPs:

(P): Minimize $0x$
s.t. $Ax = b$
$x \geq 0$

(D): Maximize $b^t y$
s.t. $A^t y \leq 0$

or, equivalently,

Maximize $y^t b$
s.t. $y^t A \leq 0$

(These LP problems have interesting characteristics:

- *every feasible solution to (P) is optimal*
- *the value $y=0$ is feasible in problem (D)*

First we will prove that if statement (i) is true, i.e.

$$\text{if } \; y^tA \le 0 \text{ for some } y, \text{ then } y^tb \le 0$$

then statement (ii) must also be true , i.e.,

$$\text{the system } \; Ax=b,\ x \ge 0 \; \text{ is feasible}$$

If statement (i) is true, then since $y=0$ is feasible in (D) with objective value 0, it must be optimal for (D) [since (i) says that every feasible solution of (D) has objective value no greater than zero].

The Fundamental Duality Theorem then implies that problem (P) is feasible, which is simply statement (ii) above.

We next want to prove that if statement (ii) is true, i.e.,

the system $Ax=b,\ x\geq 0$ is feasible

then statement (i) must also be true, i.e.,

if $y^tA\leq 0$ for some y, then $y^tb\leq 0$

Suppose that $Ax=b$ for some $x\geq 0$, and $y^tA \leq 0$. *(We need to show that* $y^tb\leq 0$ *.)*

But $y^tA \leq 0$ & $x\geq 0$ together imply that $y^tAx \leq 0$, and since $Ax=b$, that $y^tb\leq 0$. That is, statement (i) is true.

Complementary Slackness

Theorem: Suppose that $\hat{x}$ and $\hat{y}$ are feasible solutions in the primal & dual problems, respectively:

Primal:

Minimize $c^t x$
subject to:

$$Ax \geq b$$
$$x \geq 0$$

Dual:

Maximize $b^t y$
subject to:

$$A^t y \leq c$$
$$y \geq 0$$

(Complementary Slackness, cont'd.)

Then $\hat{x}$ and $\hat{y}$ are each optimal in their respective problems ***if and only if:***

- whenever a constraint of one problem is slack, then the corresponding variable of the other problem is zero
- whenever a variable of one problem is positive, then the corresponding constraint of the other problem is tight.

Proof

Proof of the Complementary Slackness Theorem:

First we introduce surplus & slack variables to the primal & dual problems, respectively:

P: Min $c^t x$
s.t.
$$Ax - I u = b$$
$$x \geq 0, \; u \geq 0$$

D: Max yb
s.t.
$$yA + I v = c^t$$
$$y \geq 0, \; v \geq 0$$

Now suppose that the vector $[\hat{x}, \hat{u}]$ *is feasible in P, and that* $[\hat{y}, \hat{v}]$ *is feasible in D.*

(Proof of Complementary Slackness, cont'd.):

Consider the difference:

$$c^t\hat{x} - \hat{y}\,b = \underbrace{(\hat{y}A + I\hat{v})}_{c^t}\hat{x} - \hat{y}\underbrace{(A\hat{x} - I\hat{u})}_{b}$$

$$= \hat{y}\,A\hat{x} + \hat{v}\,\hat{x} - \hat{y}\,A\hat{x} + \hat{y}\,\hat{u}$$

$$= \hat{v}\,\hat{x} + \hat{y}\,\hat{u}$$

(1.) Suppose that $[\hat{x},\hat{u}]$ *and* $[\hat{y},\hat{v}]$ *are both optimal in their respective problems, i.e.,* $c^t\hat{x} = \hat{y}b$.

Then $\hat{v}\,\hat{x} + \hat{y}\,\hat{u} = 0$

That is, $\sum_{j=1}^{n} \hat{v}_j\hat{x}_j + \sum_{i=1}^{m} \hat{y}_i\hat{u}_i = 0$

(Proof of Complementary Slackness, cont'd):

Since each of the factors in each term $\hat{v}_j \hat{x}_j$ *and* $\hat{y}_i \hat{u}_i$ *are nonnegative, each term is nonnegative.*

And because the sum of all terms is zero, it is clear that each term must be zero, i.e.,

$$\sum_{j=1}^{n} \hat{v}_j \hat{x}_j + \sum_{i=1}^{m} \hat{y}_i \hat{u}_i = 0 \implies \hat{v}_j \hat{x}_j = 0 \text{ for all } j=1,\ldots n \quad \& \quad \hat{y}_i \hat{u}_i = 0 \text{ for all } i=1,\ldots m$$

$$\hat{v}_j \hat{x}_j = 0 \implies \text{either } \hat{v}_j = 0 \text{ or } \hat{x}_j = 0$$

$$\hat{y}_i \hat{u}_i = 0 \implies \text{either } \hat{y}_i = 0 \text{ or } \hat{u}_i = 0$$

(Proof of Complementary Slackness, cont'd):

But $\hat{v}_j \hat{x}_j = 0 \implies$ *either* $\hat{v}_j = 0$ *or* $\hat{x}_j = 0$

i.e., when dual constraint #j is slack, the corresponding primal variable $\hat{x}_j$ *must be zero.*

and

when the primal variable $\hat{x}_j$ *is positive, then the corresponding dual constraint must be tight.*

(Proof of Complementary Slackness, cont'd.):

And $\hat{y}_i \hat{u}_i = 0 \implies$ *either* $\hat{y}_i = 0$ *or* $\hat{u}_i = 0$

i.e., when primal constraint #i is slack ($\hat{u}_i > 0$ *), then the corresponding dual variable (* $\hat{y}_i$ *) must be zero*

and

when a dual variable ($\hat{y}_i$ *) is positive, the corresponding primal constraint must be tight (so that* $\hat{u}_i = 0$ *).*

So optimality implies that complementary slackness is satisfied.

(Proof of Complementary Slackness, cont'd):

The converse is also true: if complementary slackness is satisfied, then the solutions must be optimal, since

$$c^t \hat{x} - \hat{y}\, b = \sum_{j=1}^{n} \hat{v}_j \hat{x}_j + \sum_{i=1}^{m} \hat{y}_i \hat{u}_i$$

and so, if each term is zero, the sum must be zero, i.e.,

$$c^t \hat{x} - \hat{y}\, b = 0 \quad \Longrightarrow \quad c^t \hat{x} = \hat{y}\, b$$

which, according to the Weak Duality Theorem, means that $\hat{x}$ *and* $\hat{y}$ *must both be optimal in their respective problems.*

Q.E.D.