Optimality of Hungarian Algorithm

The *Linear Assignment Problem* and its dual:

Primal Problem:

$$Min \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} X_{ij}$$

$$s.t. \quad \sum_{j=1}^{n} X_{ij} = 1 \quad \forall i = 1, 2, \ldots n$$

$$\sum_{i=1}^{n} X_{ij} = 1 \quad \forall j = 1, 2, \ldots n$$

$$X_{ij} \geq 0 \quad \forall i \,\&\, j$$

Dual Problem:

$$Max \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$$

$$s.t. \quad u_i + v_j \leq C_{ij} \quad \forall i \,\&\, j$$

$$(u \;\&\; v \text{ are unrestricted in sign})$$

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Complementary Slackness Theorem:

Suppose that $\hat{x}$ and $\hat{y}$ are *feasible* solutions in the primal (***P***) and dual (***D***) problems, respectively:

$$\textbf{P:}\quad \begin{aligned} \min\ & cx \\ s.t.\ & Ax \geq b \\ & x \geq 0 \end{aligned} \qquad\qquad \textbf{D:}\quad \begin{aligned} \max\ & b^T y \\ s.t.\ & A^T y \leq c^T \\ & y \geq 0 \end{aligned}$$

Then $\hat{x}$ and $\hat{y}$ are each *optimal* in their respective problems ***P*** and ***D*** ***if and only if***:

- whenever a constraint of one problem is slack, then the corresponding variable of the other problem is zero;
- whenever a variable of one problem is positive, then the corresponding constraint in the other problem must be tight.

The *Complementary Slackness Theorem* can be used to prove that any solution found by the Hungarian algorithm must be *optimal*:

a. Let $\hat{X}$ be the (*feasible*) solution found by the Hungarian method.

b. Let $\hat{u}_i$ and $\hat{v}_j$ be the cumulative quantities subtracted from row *i* and column *j*, respectively.

Then the contents of cell *(i,j)* of the current matrix in the Hungarian algorithm is

$$C_{ij} - \hat{u}_i - \hat{v}_j \geq 0 \quad \Rightarrow \quad \hat{u}_i + \hat{v}_j \leq C_{ij}$$

that is, $(\hat{u}, \hat{v})$ will be *feasible* in the dual problem.

Furthermore, we have chosen $\hat{X}$ so that

c. $\hat{X}_{ij} > 0 \ \ (i.e., = 1) \quad \Rightarrow \quad C_{ij} - \hat{u}_i - \hat{v}_j = 0$

and

d. $C_{ij} - \hat{u}_i - \hat{v}_j > 0 \quad \Rightarrow \quad \hat{X}_{ij} = 0$

The remaining conditions of the *Complementary Slackness Theorem* are satisfied trivially, since

e. there are no slack constraints in the primal

f. whether $\hat{u}_i$ or $\hat{v}_j$ are positive or zero, the corresponding constraint (an equation) must be tight.

Since $\hat{X}$ and $(\hat{u},\hat{v})$ are both feasible and satisfy all conditions of the *Complementary Slackness Theorem*, they must therefore both be *optimal* solutions!

Furthermore, the Fundamental Duality Theorem of LP implies that if $\hat{X}$ and $(\hat{u},\hat{v})$ are optimal, then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}\hat{X}_{ij} \quad = \quad \sum_{i=1}^{n}\hat{u}_i + \sum_{j=1}^{n}\hat{v}_j$$

i.e., the optimal cost of the assignment problem may be evaluated either by

summing C_{ij} for which $\hat{X}_{ij} = 1$

or summing the dual variables

(if you have kept a cumulative record of them.)