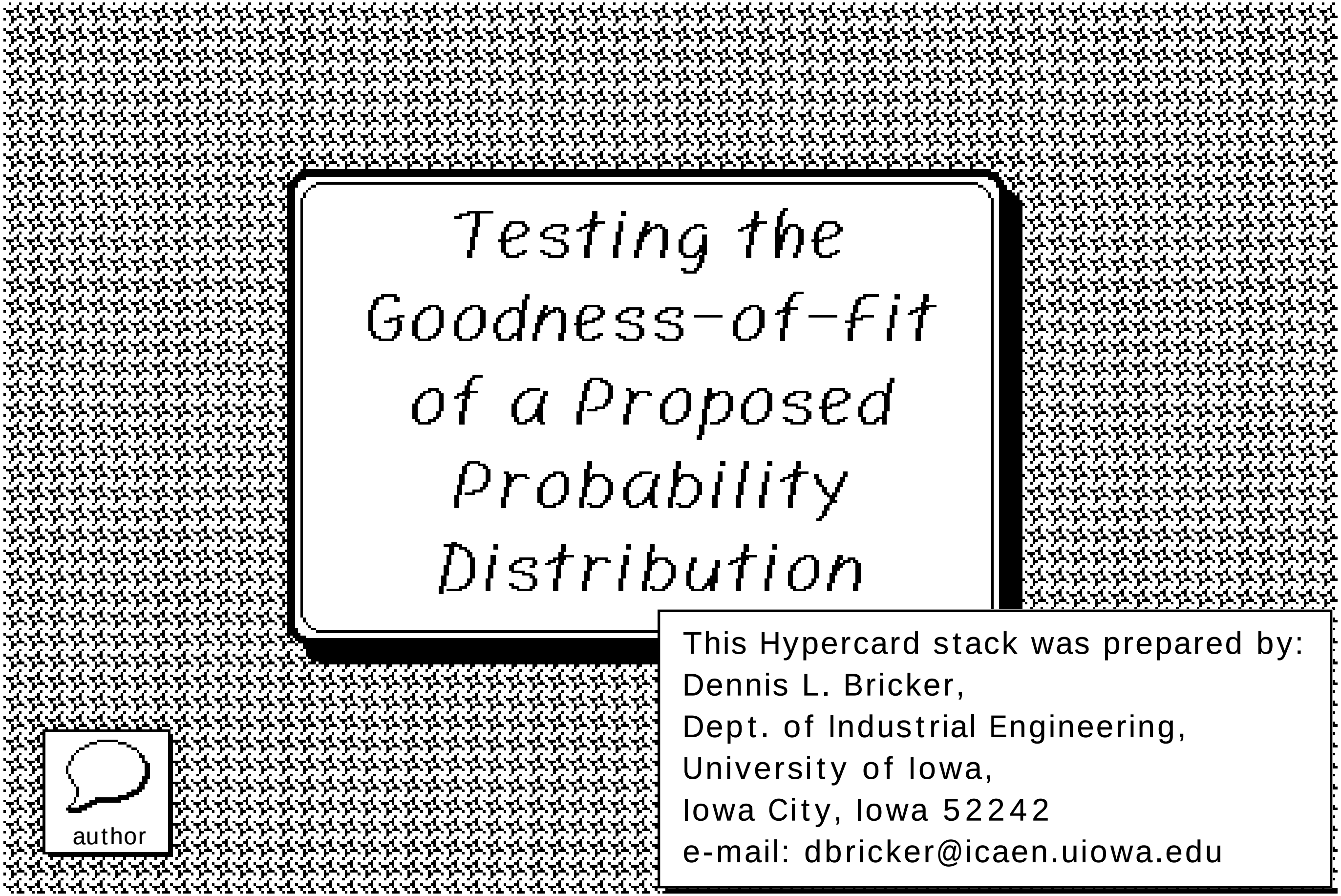

Testing the Goodness-of-Fit of a Proposed Probability Distribution

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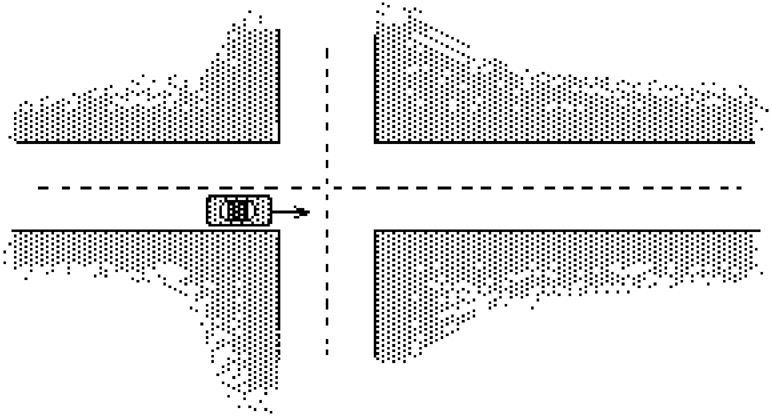
Chi-square Distribution

The *sum* of the *squares* of ν independent N(0,1) random variables has the ***chi-square*** (χ^2) distribution.

$$f_Y(y) = \frac{\frac{1}{2}\left(y/2\right)^{(\nu/2)-1} e^{-y/2}}{\Gamma(\nu/2)}, \quad y \geq 0 \qquad \mu_Y = \nu,\ \sigma_Y^2 = 2\nu$$

The parameter ν is referred to as "degrees of freedom"

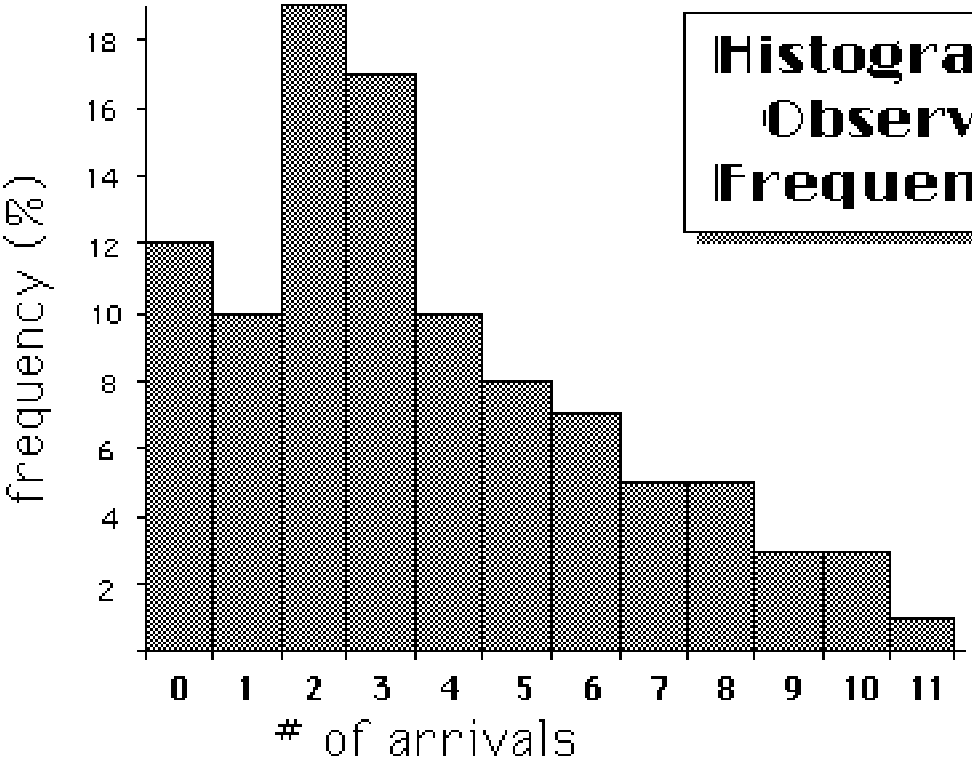
Example The number of eastbound vehicles arriving at an intersection in a 5-minute period between 7:00am & 7:05am was monitored for 5 workdays over a 20-week period (100 observations).



#arrivals	frequency
0	12
1	10
2	19
3	17
4	10
5	8
6	7
7	5
8	5
9	3
10	3
11	1

On 12 of the 100 days, no arrivals were noted; on 10 of the days, 1 arrival was noted; etc.

Histogram of Observed Frequencies



Suppose that p_i = probability of i arrivals between 7:00am and 7:05am each day

Then in a sample of size N=100, the number of days that we observe i arrivals will have the ***Binomial*** Distribution with parameters N and p_i:

$$P\{X = x\} = \frac{N!}{x!(N-x)!} p_i^x (1-p_i)^{N-x}$$

= probability that i arrivals are observed on **x** out of N days

It was guessed that the arrival process is Poisson, i.e., the time between arrivals has an ***Exponential*** distribution,

and the number of arrivals in the 5-minute interval has a ***Poisson*** distribution:

$$P\{N_t = x\} = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \quad \text{with mean } \lambda t$$

The average of the observed values is 3.64, so we estimate the arrival rate to be

$$\lambda = {}^{3.64}/_{5 \text{ min.}} = 0.728/\text{min.}$$

$$P\{N_t = x\} = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

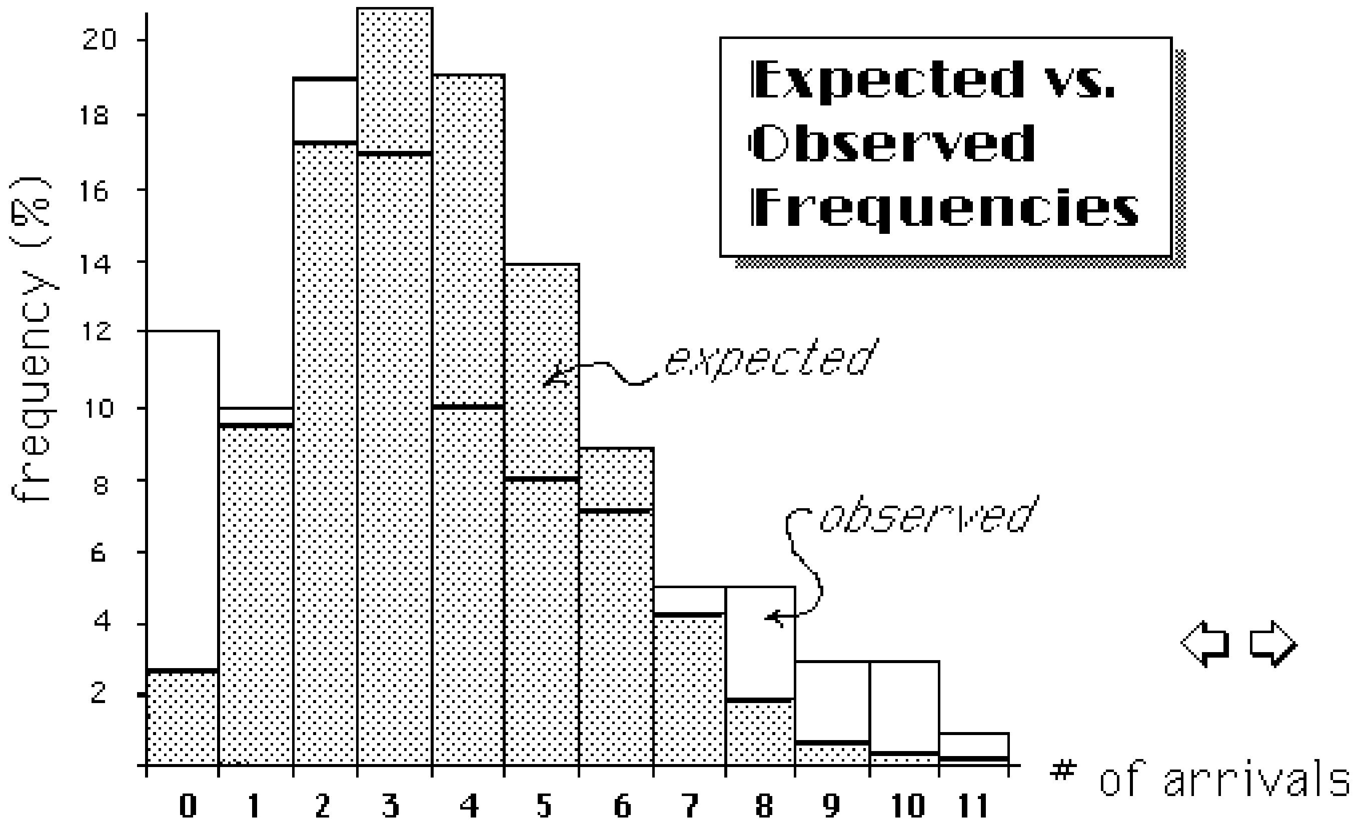
$$P\{N_5 = x\} = \frac{e^{-3.64}3.64^x}{x!}$$

For example, the probability of observing 0 arrivals is, according to this probability model,

$$P\{N_5 = 0\} = 0.026,$$

so that the expected number of observations of 0 arrivals during the 100 days is 0.026x100 = 2.6.

Expected vs. Observed Frequencies



Goodness-of-Fit Test

Suppose that we have N observations of a discrete random variable X, with O_i = # of observations of the i^{th} possible value.

Let p_i = probability of observing this i^{th} value.

Then

O_i has the ***binomial*** distribution,

with ***expected value*** $E_i = Np_i$

and ***variance*** $= Np_i(1-p_i)$.

By the Central Limit Theorem, $$\frac{O_i - \overset{\textit{mean}}{Np_i}}{\underbrace{\sqrt{Np_i(1-p_i)}}_{\textit{standard deviation}}}$$

has approximately a N(0,1) distribution, so that the sum of the squares

$$D' = \sum_{i=1}^{k} \frac{(O_i - Np_i)^2}{Np_i(1-p_i)}$$

has approximately a χ^2 distribution.

Owing to the lack of independence among the O_i,

$$\text{i.e., } O_i = N - \sum_{j \neq i} O_j$$

the number of "degrees of freedom" of the variable is not k, but k−1, and each term is decreased by a factor of $(1-p_i)$:

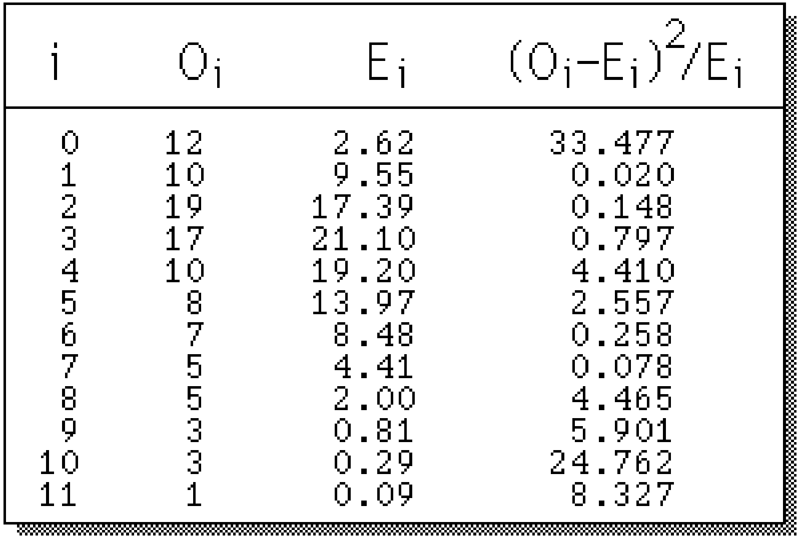
$$D = \sum_{i=1}^{k} \frac{(O_i - Np_i)^2}{Np_i(1-p_i)}(1-p_i) = \sum_{i=1}^{k} \frac{(O_i - Np_i)^2}{Np_i}$$

$$= \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

i	O_i	E_i
0	12	2.6
1	10	9.6
2	19	17.4
3	17	21.1
4	10	19.2
5	8	14.0
6	7	8.5
7	5	4.4
8	5	2.0
9	3	0.8
10	3	0.3
11	1	0.1

The observed frequencies of the number of arrivals differs from the expected number for every i.

Are the differences between O_i and E_i large enough to justify rejecting the Poisson distribution with $\lambda = 0.728$/min. ?



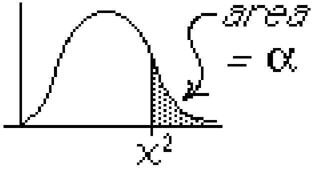
i	O_i	E_i	$(O_i-E_i)^2/E_i$
0	12	2.62	33.477
1	10	9.55	0.020
2	19	17.39	0.148
3	17	21.10	0.797
4	10	19.20	4.410
5	8	13.97	2.557
6	7	8.48	0.258
7	5	4.41	0.078
8	5	2.00	4.465
9	3	0.81	5.901
10	3	0.29	24.762
11	1	0.09	8.327

sum: 85.206

Chi-square Tables

From tables of the chi-square distribution, we can find the value $\chi^2_{\alpha,\nu}$ such that

$$P\{D \geq \chi^2_{\alpha,\nu}\} = \alpha$$



After observing a sample of the random variable X, we can compute the observed value of D and compare it to the tabulated value of $\chi^2_{\alpha,\nu}$.

(ν = degrees of freedom)

chi-square table

Chi-square Table

degrees of freedom	$P\{D > \chi^2\}$		
	10%	5%	1%
1	2.706	3.841	6.635
2	4.605	5.991	9.210
3	6.251	7.815	11.345
4	7.779	9.488	13.277
5	9.236	11.070	15.086
6	10.645	12.592	16.812
7	12.017	14.067	18.475
8	13.362	15.507	20.090
9	14.684	16.919	21.666
10	15.987	18.307	23.209
11	17.275	19.675	24.725
12	18.549	21.026	26.217
13	19.812	22.362	27.688
14	21.064	23.685	29.141
15	22.307	24.996	30.578
16	23.542	26.296	32.000

For our example, the number of cells in the histogram is 12, and the number of parameters estimated from the data is 1, so that the number of "degrees of freedom" is 12 - 1 - 1 = 10.

For $\alpha = 5\%$, $\chi^2_{\alpha,10} = 18.307$ from the table.
That is, the probability that the observed value of D exceeds 18.307, given the Poisson model with mean value 3.64 arrivals, is only 5%.

Since we calculated the value D=85.2, this would lead us to believe that the assumed model is ***not*** valid!

It has been recommended, however, that the observations be "lumped" together so that each cell contains at least 5 to 10 observations.

i	O_i		E_i	
0	12	22	2.62	12.2
1	10		9.55	
2	19		17.39	
3	17		21.10	
4	10		19.20	
5	8		13.97	
6	7		8.48	
7	5	17	4.41	7.6
8	5		2.00	
9	3		0.81	
10	3		0.29	
11	1		0.09	

i	O_i	E_i	$(O_i-E_i)^2/E_i$
≤ 1	22	12.181	7.914
2	19	17.391	0.148
3	17	21.101	0.797
4	10	19.202	4.410
5	8	13.979	2.557
6	7	8.480	0.258
≥ 7	17	7.621	11.541

Sums: 100 99.959 27.628

chi-square table

The degrees of freedom are now 7-1-1=5, and for $\alpha = 5\%$, the table gives

$$\chi^2_{\alpha,5} = 11.070$$

Again, since $D > \chi^2_{\alpha,5}$, we would reject the Poisson model with mean 3.64.

Gap (sec.)	# gaps	Gap (sec.)	# gaps	Gap (sec.)	# gaps
0-1	18	11-12	12	22-23	1
1-2	25	12-13	6	23-24	0
2-3	21	13-14	3	24-25	1
3-4	13	14-15	3	25-26	0
4-5	11	15-16	3	26-27	1
5-6	15	16-17	6	27-28	1
6-7	16	17-18	4	28-29	1
7-8	12	18-19	3	29-30	2
8-9	11	19-20	3	30-31	1
9-10	11	20-21	1		
10-11	8	21-22	1		

Example

Observed gaps in traffic total observations=214

The average of the observed values of the gaps is

$$\frac{0.5\times 18 + 1.5\times 25 + 2.5\times 21 + \ldots + 29.5\times 2 + 30.5\times 1}{214}$$

= 7.66354 seconds. If the process is Poisson, i.e., if the gaps have an exponential distribution, the estimated arrival rate is

$$\lambda = \frac{1}{7.66354 \text{ sec.}} = 0.130488 \text{ /sec.}$$

(midpoint of each cell was used in computing average.)

Gap	O_j	p_j	E_j	$\lvert O_j-E_j \rvert$	$(O_j-E_j)^2/E_j$
0–1	18	0.122333	26.1793	8.17926	2.55547
1–2	25	0.107368	22.9767	2.02333	0.178175
2–3	21	0.094233	20.1659	0.834134	0.0345028
3–4	13	0.0827052	17.6989	4.69892	1.24752
4–5	11	0.0725876	15.5338	4.53375	1.32324
5–6	15	0.0637078	13.6335	1.36654	0.136973
6–7	16	0.0559142	11.9656	4.03436	1.36023
7–8	12	0.0490741	10.5018	1.49815	0.21372
8–9	11	0.0430707	9.21713	1.78287	0.344862
9–10	11	0.0378017	8.08957	2.91043	1.0471
10–12	20	0.062296	13.3313	6.66866	3.33583
12–13	6	0.0255565	5.46909	0.530915	0.0515389
13–16	9	0.0593941	12.7103	3.71033	1.0831
16–19	13	0.0401543	8.59303	4.40697	2.26013
19–24	6	0.0401612	8.5945	2.5945	0.783223
24–∞	7	0.043643	9.33961	2.33961	0.58608

$214xp_j$

D=16.5417

k=16 intervals, and we have used the data to estimate 1 parameter (λ), so the number of "degrees of freedom" is 16−1−1=14.

According to the table, for $\alpha = 5\%$, $\chi^2_{0.05,14} = 23.685$ i.e.,

$$P\{D \geq 23.685\} = 5\%$$

and since the observed value of D is 16.54, we do not reject the hypothesis that the traffic gap has an exponential distribution with $\lambda = 0.130/\text{sec}$.

chi-square table

In fact, $P\{D \geq 16.5 \} \doteq 0.284$

That is, there is about a 28% probability that the value of D would exceed the observed value.

Example *"Incredibly good" data*

In a statistics laboratory, students were asked to draw samples of 4 from a bowl containing red and black balls in equal proportions, with the drawn balls being returned into the bowl after each test. 160 samples were to be drawn, and the results reported. Group A reported the following:

# red balls	0	1	2	3	4
# observations	9	40	59	41	11
# expected	10	40	60	40	10

i	0	1	2	3	4
O_i	9	40	59	41	11
E_i	10	40	60	40	10
O_i-E_i	1	0	1	1	1
$(O_i-E_i)^2/E_i$	$\frac{1}{10}$	0	$\frac{1}{60}$	$\frac{1}{40}$	$\frac{1}{10}$

Sum: 0.24

The number of degrees of freedom is 5−1=4, and the table gives, for $\alpha = 99\,\%$, $\chi^2_{0.99,4} = 0.297$. Therefore, the probability that the students actually observed the results reported is less than 1%, leading the prof. to suspect that the lab report was faked!

Time between arrivals (min.)	Frequency
1	17
2	15
3	12
4	10
5	9
6	8
7	6
8	6
9	4
10	3
11	3
12	2
13	2
14	1
15	1
16	1

How good a fit to the observed data is the exponential probability distribution with mean interarrival time equal to 5 minutes, i.e.,

$$\lambda = {}^{1}\!/_{5\text{min.}} = 0.2/\text{min.}\ ?$$

The probability that T_1, the next arrival time, falls in each interval is found using the *Exponential CDF*:

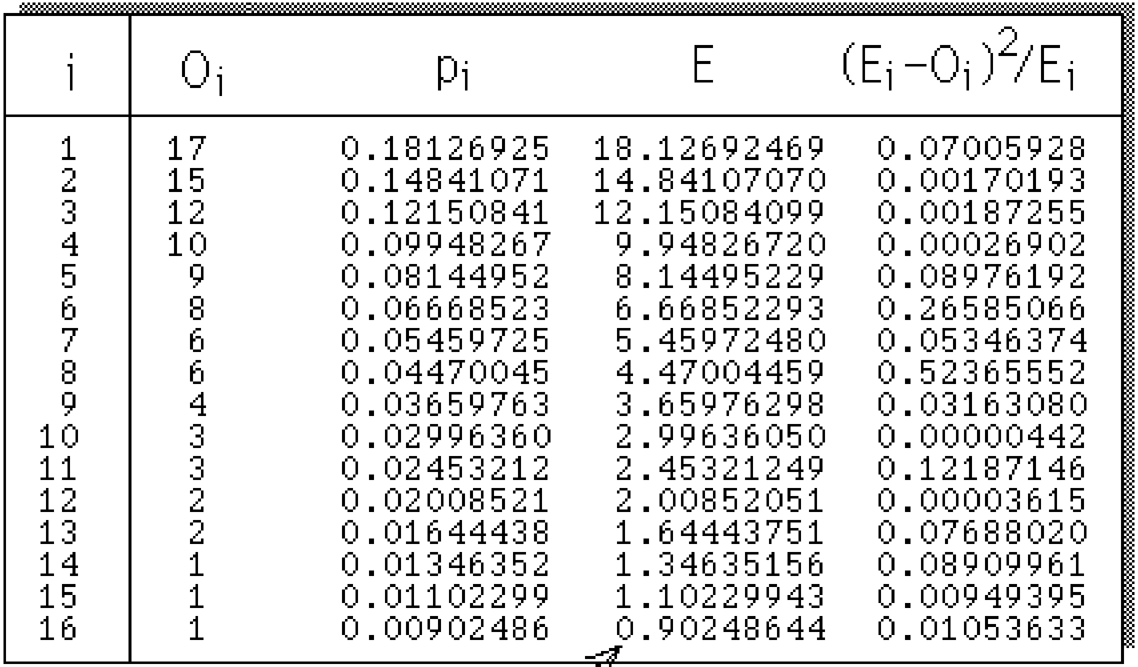
$$P\{T_1 \le t\} = F(t) = 1 - e^{-\lambda t} = 1 - e^{-0.2t}$$

$$P\{t_0 \le T_1 \le t_1\} = F(t_1) - F(t_0) = e^{-\lambda t_0} - e^{-\lambda t_1}$$

For example,

$$\begin{aligned} P\{0 \le T_1 \le 1\} = F(1) - F(0) &= e^{0} - e^{-0.2} \\ &= 1 - 0.8187307 \\ &= 0.181269 \end{aligned}$$

$$\begin{aligned} P\{1 \le T_1 \le 2\} = F(2) - F(1) &= e^{-0.2} - e^{-0.4} \\ &= 0.8187307 - 0.67032 \\ &= 0.148411 \end{aligned}$$



i	O_i	p_i	E	$(E_i - O_i)^2/E_i$
1	17	0.18126925	18.12692469	0.07005928
2	15	0.14841071	14.84107070	0.00170193
3	12	0.12150841	12.15084099	0.00187255
4	10	0.09948267	9.94826720	0.00026902
5	9	0.08144952	8.14495229	0.08976192
6	8	0.06668523	6.66852293	0.26585066
7	6	0.05459725	5.45972480	0.05346374
8	6	0.04470045	4.47004459	0.52365552
9	4	0.03659763	3.65976298	0.03163080
10	3	0.02996360	2.99636050	0.00000442
11	3	0.02453212	2.45321249	0.12187146
12	2	0.02008521	2.00852051	0.00003615
13	2	0.01644438	1.64443751	0.07688020
14	1	0.01346352	1.34635156	0.08909961
15	1	0.01102299	1.10229943	0.00949395
16	1	0.00902486	0.90248644	0.01053633

Sum: D=1.34619

$E_i = 100 p_i$

From a χ^2-probability table, it appears that

$$P\{D \leq 1.4\} = 0.001\%$$

That is, the fit appears "too good"!

(One should be suspicious that the recorded data was fabricated or "doctored"!)

chi-square table

Service Time (min.)	Frequency
1	51
2	23
3	12
4	7
5	4
6	2
7	1

How good a fit to the observations is the exponential distribution with mean service time of 2 minutes, i.e.,

$$\mu = {}^{1}\!/_{2\text{ min.}} = 0.5/\text{min.}?$$

i	O_i	p_i	E_i	$(E_i-O_i)^2/E_i$
1	51	0.393469	39.3469	3.4512
2	23	0.238651	23.8651	0.0313611
3	12	0.144749	14.4749	0.423164
4	7	0.0877949	8.77949	0.360679
5	4	0.0532503	5.32503	0.329707
6	2	0.0322979	3.22979	0.468262
7	1	0.0195897	1.95897	0.469441
8	0	0.0301974	3.01974	3.01974

Sum: D=8.55355

degrees of freedom = 8 − 1 − 1 = 6.

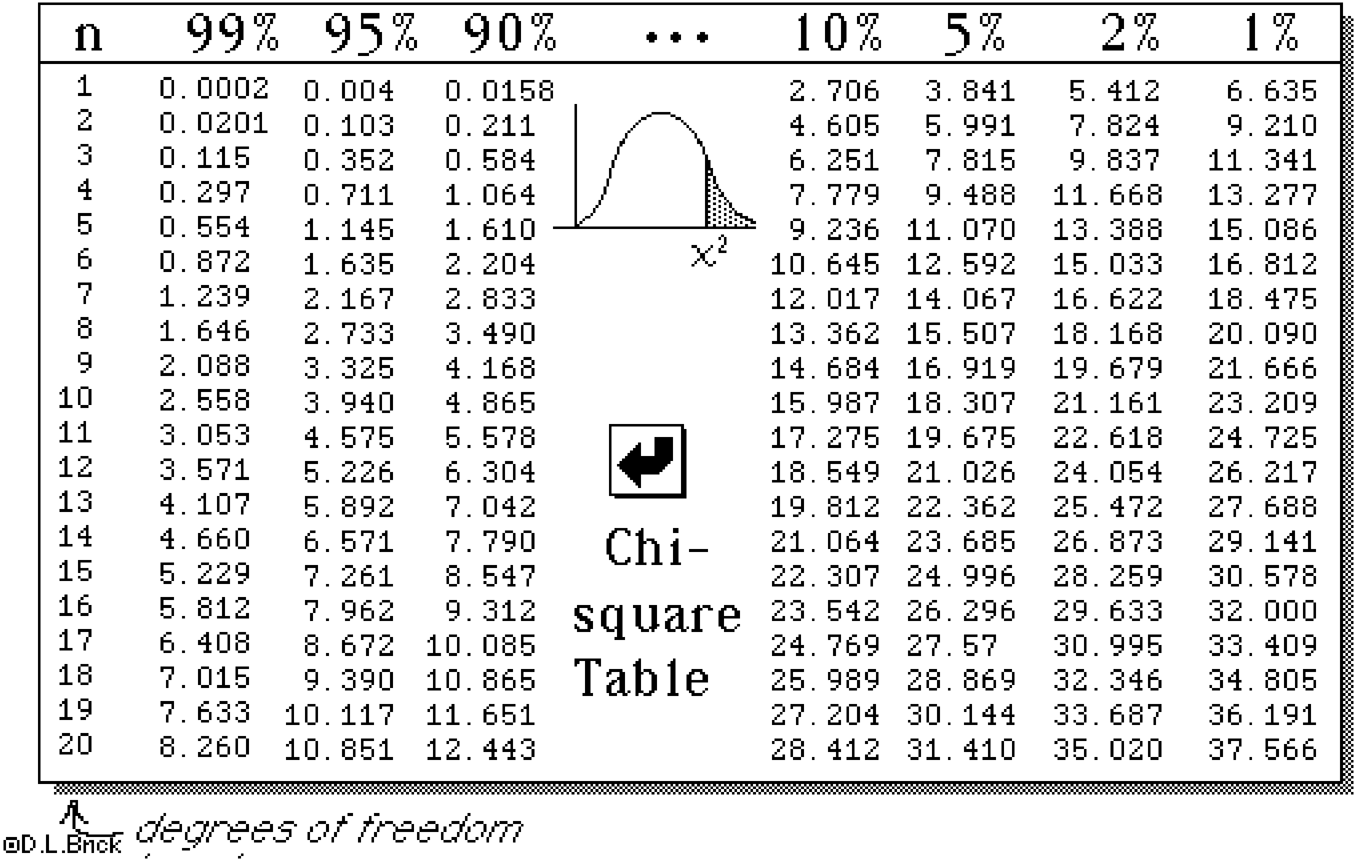
What is χ^2 such that $P\{D > \chi^2\} = 5\%$?

From a χ^2-probability table,

$$P\{D > \chi^2\} = 5\% \qquad \text{for } \chi^2 = 12.59$$

The observed value of D=8.55 is, therefore, not so large as to indicate a poor fit of the model to the observed data.

chi-square table



n	99%	95%	90%	•••	10%	5%	2%	1%
1	0.0002	0.004	0.0158		2.706	3.841	5.412	6.635
2	0.0201	0.103	0.211		4.605	5.991	7.824	9.210
3	0.115	0.352	0.584		6.251	7.815	9.837	11.341
4	0.297	0.711	1.064		7.779	9.488	11.668	13.277
5	0.554	1.145	1.610		9.236	11.070	13.388	15.086
6	0.872	1.635	2.204		10.645	12.592	15.033	16.812
7	1.239	2.167	2.833		12.017	14.067	16.622	18.475
8	1.646	2.733	3.490		13.362	15.507	18.168	20.090
9	2.088	3.325	4.168		14.684	16.919	19.679	21.666
10	2.558	3.940	4.865		15.987	18.307	21.161	23.209
11	3.053	4.575	5.578		17.275	19.675	22.618	24.725
12	3.571	5.226	6.304		18.549	21.026	24.054	26.217
13	4.107	5.892	7.042		19.812	22.362	25.472	27.688
14	4.660	6.571	7.790		21.064	23.685	26.873	29.141
15	5.229	7.261	8.547		22.307	24.996	28.259	30.578
16	5.812	7.962	9.312		23.542	26.296	29.633	32.000
17	6.408	8.672	10.085		24.769	27.57	30.995	33.409
18	7.015	9.390	10.865		25.989	28.869	32.346	34.805
19	7.633	10.117	11.651		27.204	30.144	33.687	36.191
20	8.260	10.851	12.443		28.412	31.410	35.020	37.566

degrees of freedom

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