Generalized Assignment Problem

"Strong" Lagrangian Relaxation

In this algorithm, we relax the semi-assignment constraints,

$$\sum_{i=1}^{m} X_{ij} = 1 \qquad \text{for all } j = 1, 2, \ldots n$$

while maintaining the machine resource constraints,

$$\sum_{j=1}^{n} A_{ij} X_{ij} \le b_i \qquad \text{for all } i = 1, 2, \ldots m$$

Sample Generalized Assignment Problem

Costs

Machine i	Job 1	Job 2	Job 3	Job 4	Job 5	Job 6	Job 7	Job 8	Job 9
1	10	10	18	25	21	23	24	12	18
2	21	20	12	17	17	14	23	25	24
3	10	22	12	22	14	25	25	23	13
4	20	12	21	16	17	19	17	11	15

Resources Used

Machine i	Job 1	Job 2	Job 3	Job 4	Job 5	Job 6	Job 7	Job 8	Job 9	Available
1	11	7	22	9	15	12	13	10	10	38
2	12	16	20	10	10	23	24	21	20	44
3	9	20	7	23	5	18	10	13	7	19
4	8	23	13	25	13	13	6	14	25	61

Lagrangian Dual of GAP

(Solved via subgradient optimization)
Multiple choice constraints are relaxed.
Lambda = 0.9
Target Z* = 117 *(determined previously by a heuristic algorithm!)*

At each iteration we take a step g in the subgradient direction, where

$$\gamma_j = 1 - \sum_{i=1}^{m} X_{ij}$$

The sum $\sum_{i=1}^{m} X_{ij}$ is, of course, the number of machines to which job j has been assigned.

The step is computed by the formula:

$$w' = w + \lambda \frac{\bar{Z} - \underline{Z}}{\|\gamma\|^2} \gamma$$

where

w is the current vector of Lagrangian multipliers,

λ is the *stepsize parameter*, $0 < \lambda \leq 2$

$\bar{Z}$ is the current *target* value (≥Z*)

and

$\underline{Z}$ is the current value of the Lagrangian dual, i.e., the lower bound provided by the Lagrangian relaxation.

Iteration # 1

```
Current multipliers:
 i:    1  2  3  4  5  6  7  8  9
w[i]: 20 20 18 22 17 23 24 23 18
```
(second-largest C_{ij})

Lagrangian relaxation coefficients $C_{ij} - w_j$

job	1	2	3	4	5	6	7	8	9
1	¯10	¯10	0	3	4	0	0	¯11	0
2	1	0	¯6	¯5	0	¯9	¯1	2	6
3	¯10	2	¯6	0	¯3	2	1	0	¯5
4	0	¯8	3	¯6	0	¯4	¯7	¯12	¯3

By the choice of the initial multipliers, at most one coefficient is positive in each column.

Solution of Lagrangian relaxation (knapsack solutions):

```
 Machine #1:  items 1 2 8
 Machine #2:  items 3 6
 Machine #3:  items 1 3
 Machine #4:  items 2 6 7 8
*** Dual value is 92 ***
```

<u># of times jobs are assigned:</u>

i:	1	2	3	4	5	6	7	8	9
# :	2	2	2	0	0	2	1	2	0

Applying heuristic to X:
Feasible solution has cost 117

Subgradient of Dual Objective is ¯1 ¯1 ¯1 1 1 ¯1 0 ¯1 1 (γ)
Stepsize is 2.813

$$\lambda \frac{\overline{Z} - \underline{Z}}{\|\gamma\|^2} = 0.9 \times \frac{117 - 92}{8} = 2.813$$

Thus, the updated value of the multiplier for job # 1 is

$$w_1' = w_1 + 2.813\gamma_1 = 20 - 2.813 = 17.187$$

Iteration # 2

Current multipliers:

i:	1	2	3	4	5	6	7	8	9
w[i]:	17.19	17.19	15.19	24.81	19.81	20.19	24	20.19	20.81

Lagrangian relaxation coefficients

job

	1	2	3	4	5	6	7	8	9
1	¯7.188	¯7.188	2.813	0.1875	1.188	2.813	0	¯8.188	¯2.813
2	3.813	2.813	¯3.188	¯7.813	¯2.813	¯6.188	¯1	4.813	3.188
3	¯7.188	4.813	¯3.188	¯2.813	¯5.813	4.813	1	2.813	¯7.813
4	2.813	¯5.188	5.813	¯8.813	¯2.813	¯1.188	¯7	¯9.188	¯5.813

Machine #1: items 1 2 8 9
Machine #2: items 4 5 6
Machine #3: items 3 5 9
Machine #4: items 4 5 7 8

*** Dual value is 92.56 *** (Improvement: 0.5625)

# of times jobs are assigned:

i:	1	2	3	4	5	6	7	8	9
# :	1	1	1	2	3	1	1	2	2

Applying heuristic to X:
Feasible solution has cost 117

Subgradient of Dual Objective is 0 0 0 ¯1 ¯2 0 0 ¯1 ¯1
Stepsize is 3.142

Iteration # 3

Current multipliers:

i:	1	2	3	4	5	6	7	8	9
w[i]:	17.19	17.19	15.19	21.67	13.53	20.19	24	17.05	17.67

Lagrangian relaxation coefficients

job

	1	2	3	4	5	6	7	8	9
1	¯7.188	¯7.188	2.813	3.329	7.471	2.813	0	¯5.046	0.3295
2	3.813	2.813	¯3.188	¯4.671	3.471	¯6.188	¯1	7.954	6.329
3	¯7.188	4.813	¯3.188	0.3295	0.4714	4.813	1	5.954	¯4.671
4	2.813	¯5.188	5.813	¯5.671	3.471	¯1.188	¯7	¯6.046	¯2.671

Machine #1: items 1 2 8
Machine #2: items 4 6
Machine #3: items 1 9
Machine #4: items 4 6 7 8

*** Dual value is 101.6 *** (Improvement: 9.063)

# of times jobs are assigned:

i:	1	2	3	4	5	6	7	8	9
# :	2	1	0	2	0	2	1	2	1

Applying heuristic to X:
Feasible solution has cost 117

Subgradient of Dual Objective is ¯1 0 1 ¯1 1 ¯1 0 ¯1 0
Stepsize is 2.306

Iteration # 4

Current multipliers:

i:	1	2	3	4	5	6	7	8	9
w[i]:	14.88	17.19	17.49	19.36	15.83	17.88	24	14.74	17.67

<u>Lagrangian relaxation coefficients</u>

job

|	1	2	3	4	5	6	7	8	9
1|	¯4.881	¯7.188	0.5063	5.636	5.165	5.119	0	¯2.739	0.3295
2|	6.119	2.813	¯5.494	¯2.364	1.165	¯3.881	¯1	10.26	6.329
3|	¯4.881	4.813	¯5.494	2.636	¯1.835	7.119	1	8.261	¯4.671
4|	5.119	¯5.188	3.506	¯3.364	1.165	1.119	¯7	¯3.739	¯2.671

Machine #1: items 1 2 8
Machine #2: items 3 6
Machine #3: items 3 5 9
Machine #4: items 2 7 8

*** Dual value is 106.9 *** (Improvement: 5.319)

```
# of times jobs are assigned:
    i: 1 2 3 4 5 6 7 8 9
   # : 1 2 2 0 1 1 1 2 1

Applying heuristic to X:
Feasible solution has cost 117

Subgradient of Dual Objective is 0 ¯1 ¯1 1 0 0 0 ¯1 0
     Stepsize is 2.263
```

Iteration # 5

Current multipliers:

i:	1	2	3	4	5	6	7	8	9
w[i]:	14.88	14.92	15.23	21.63	15.83	17.88	24	12.48	17.67

Lagrangian relaxation coefficients

	job								
o	1	2	3	4	5	6	7	8	9
1	¯4.881	¯4.925	2.769	3.373	5.165	5.119	0	¯0.4766	0.3295
2	6.119	5.075	¯3.231	¯4.627	1.165	¯3.881	¯1	12.52	6.329
3	¯4.881	7.075	¯3.231	0.3731	¯1.835	7.119	1	10.52	¯4.671
4	5.119	¯2.925	5.769	¯5.627	1.165	1.119	¯7	¯1.477	¯2.671

Machine #1: items 1 2 8
Machine #2: items 4 6
Machine #3: items 3 5 9
Machine #4: items 2 4 7

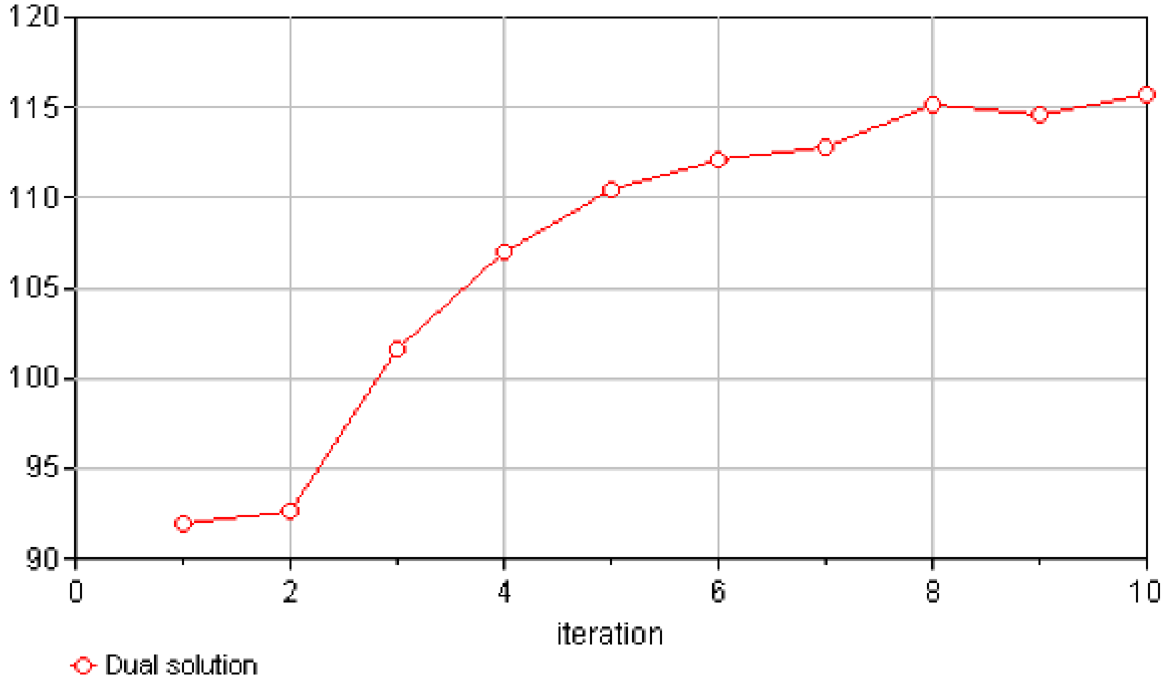
*** Dual value is 110.4 ***   (Improvement: 3.504)

```
# of times jobs are assigned:
 i: 1 2 3 4 5 6 7 8 9
# : 1 2 1 2 1 1 1 1 1

Applying heuristic to X:
Feasible solution has cost 118

Subgradient of Dual Objective is 0 ¯1 0 ¯1 0 0 0 0 0
     Stepsize is 2.948
```

… etc.



Note that the dual objective value is not monotonically increasing (although in this case, it is nearly so!)