Bayes' Rule & Decision Trees

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author

Incorporating new information in the decision tree

- Bayes' Rule
- PROTRAC, Inc. Problem
- Farmer Jones' Problem

Given

$S_1, S_2, \ldots S_n$	possible states of nature
$P\{S_i\}$	*prior* probabilities
$O_1, O_2, \ldots O_m$	possible outcomes of an experiment
$P\{O_j \mid S_i\}$	likelihood of an outcome

Calculate

$P\{S_i \mid O_j\}$	*posterior* probabilities

By the definition of conditional probability,

$$P\{S_i|O_j\} = \frac{P\{S_i \cap O_j\}}{P\{O_j\}}$$

$$\Longrightarrow \quad P\{S_i \cap O_j\} = P\{S_i|O_j\}\,P\{O_j\} = P\{O_j|S_i\}\,P\{S_i\}$$

Bayes' Rule

$$P\{S_i \cap O_j\} = P\{S_i|O_j\}\,P\{O_j\} = P\{O_j|S_i\}\,P\{S_i\}$$

$$\Rightarrow \quad P\{S_i|O_j\} = \frac{P\{O_j|S_i\}\,P\{S_i\}}{P\{O_j\}}$$

Incorporating New Information

Suppose that in the PROTRAC example, a market research study can be made before deciding which strategy (A, B, or C) to select.
The results of this study can then be used to more accurately estimate the probabilities of a "Strong" or "Weak" market.

Test results are either

- Encouraging
- Discouraging

Reliability of the market study: "The past results with our test have tended to be in the 'right direction'. Specifically, in 60% of the instances when the market has been strong, the preceding market study was 'Encouraging', while in 70% of the instances when the market has been weak, the preceding market study was 'Discouraging'."

The statement about "reliability" of the market study provides:

Conditional Probabilities:

"In 60% of the instances when the market has been strong, the preceding market study was 'encouraging' "

$P\{E \mid S\} = 60\%$

$P\{D \mid S\} = 40\%$

"In 70% of the instances when the market has been weak, the preceding market study was 'discouraging' "

$P\{E \mid W\} = 30\%$

$P\{D \mid W\} = 70\%$

Bayes' Rule can now be used to find the values for $P\{S|E\}$, $P\{S|D\}$, etc.

For example,

$$P\{S|E\} = \frac{P\{E|S\}\,P\{S\}}{P\{E\}}$$

$$= \frac{P\{E|S\}\,P\{S\}}{P\{E|S\}P\{S\} + P\{E|W\}P\{W\}}$$

$$= \frac{(0.6)(0.45)}{(0.6)(0.45) + (0.3)(0.55)}$$

$$= \frac{0.27}{0.27 + 0.165} = \frac{0.27}{0.435} = 0.621$$

$P\{E|S\} = 60\%$
$P\{D|S\} = 40\%$
$P\{E|W\} = 30\%$
$P\{D|W\} = 70\%$

prior probability of a Strong market

posterior probability of a Strong market

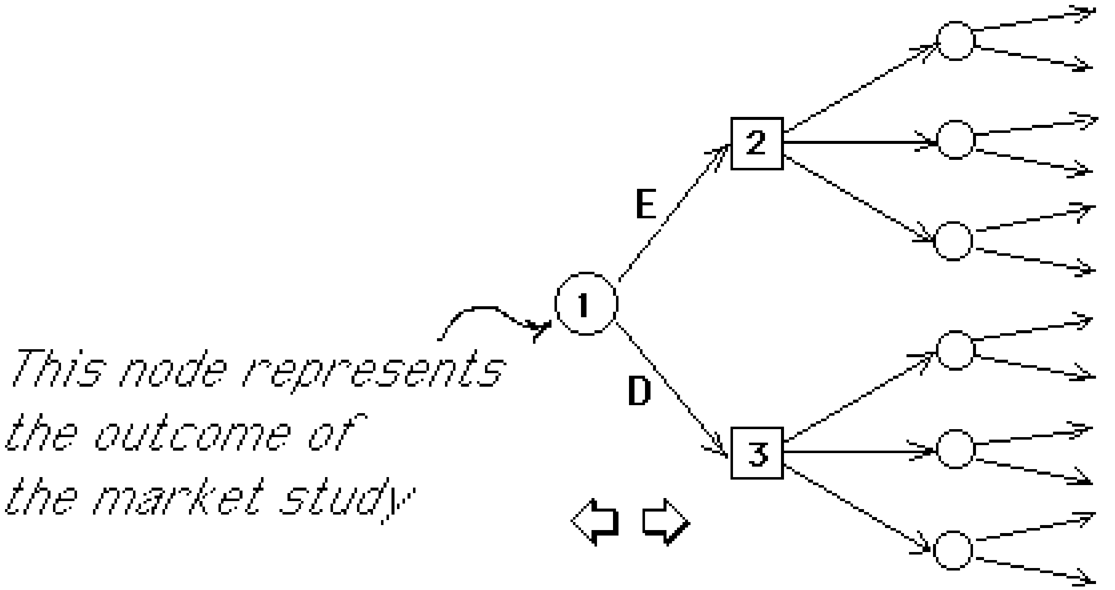
$$P\{S|E\} = \frac{P\{E|S\}\,P\{S\}}{P\{E\}}$$

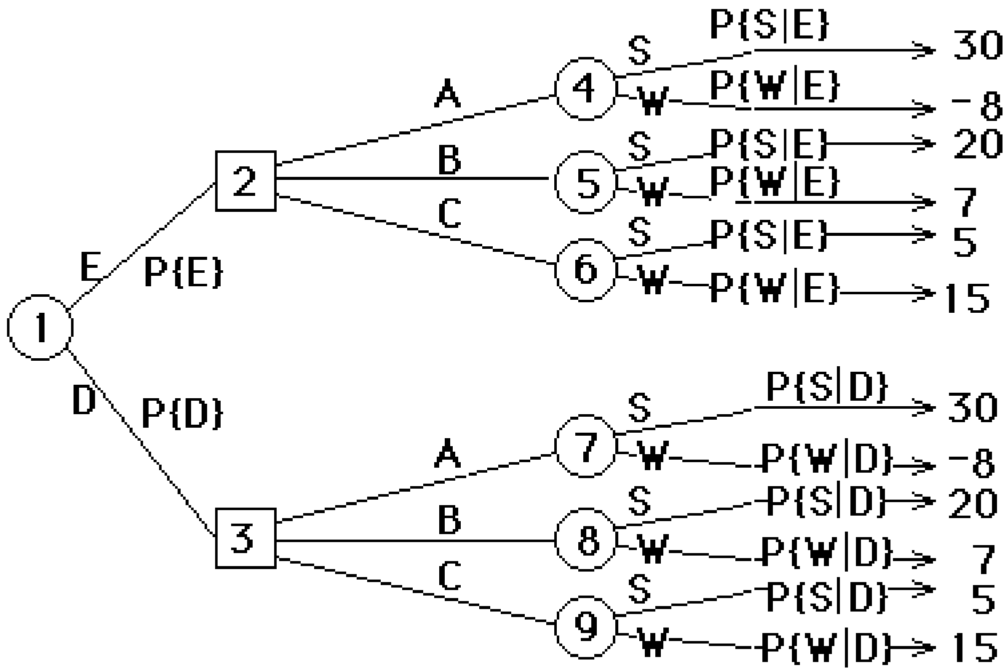
$$= \frac{(0.6)(0.45)}{(0.6)(0.45) + (0.3)(0.55)}$$

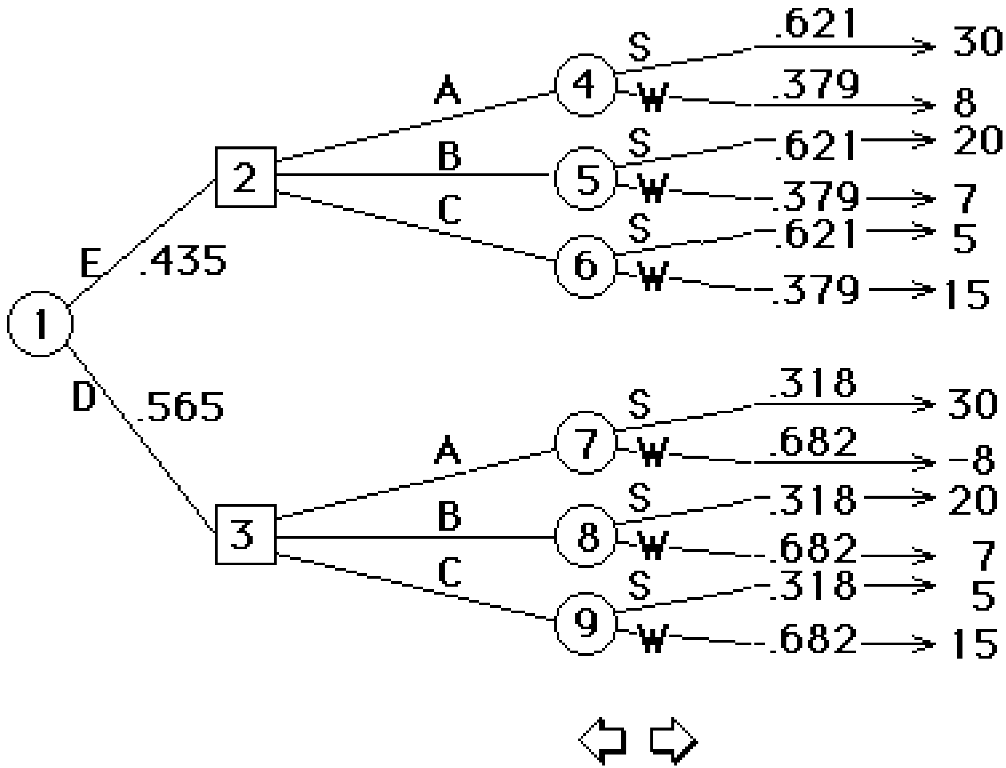
$$= \frac{0.27}{0.27 + 0.165} = \frac{0.27}{0.435} = 0.621$$

$P\{E|S\} = 60\%$
$P\{D|S\} = 40\%$
$P\{E|W\} = 30\%$
$P\{D|W\} = 70\%$

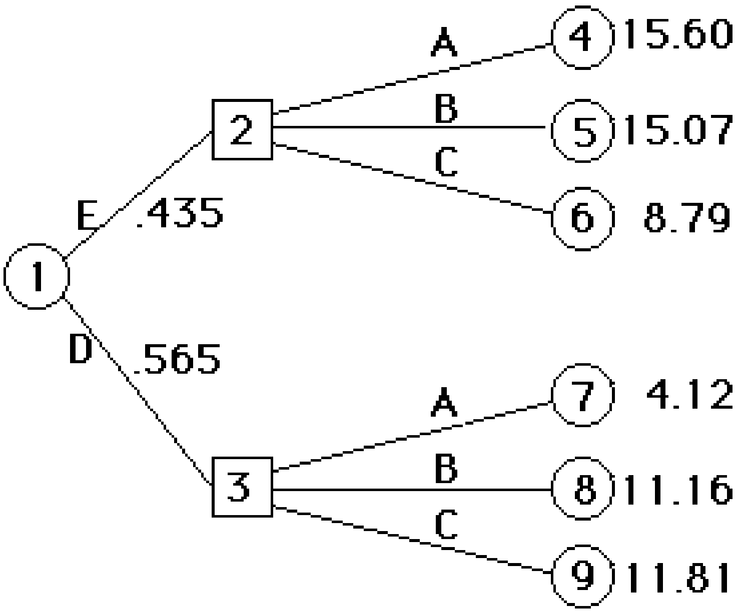
The decision tree is now drawn with the decision nodes ***following*** the (random) outcome of the market study:



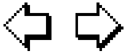

1
E
P{E}
D
P{D}
2
3
A
B
C
4
5
6
7
8
9
S
W
P{S|E}
P{W|E}
P{S|D}
P{W|D}
30
-8
20
7
5
15



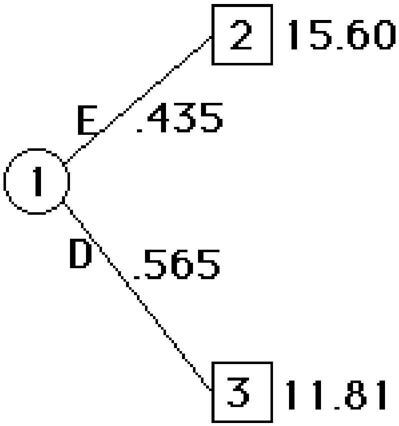
1
E .435
D .565
2
3
A
B
C
4
5
6
7
8
9
S
W
.621
.379
.318
.682
30
8
20
7
5
15
-8



"Folding back the tree"



"Folding back the tree"



"Folding back the tree"

15.6(0.435) + 11.81(0.5.65)

(1) 13.46

The maximum expected payoff which can be attained is 13.46

Expected Value of Sample Information

EVWSI: "Expected Value With Sample Information"

EVWOI: "Expected Value Without Information"

EVSI: "Expected Value of Sample Information"

EVSI = EVWSI - EVWOI

EXAMPLE

$$EVSI = EVWSI - EVWOI$$

In the "PROTRAC" decision problem,

$EVWOI = 12.85$ ← *Expected payoff with no market study*

Expected payoff using market study → $EVWSI = 13.46$

$$EVSI = 13.46 - 12.85 = 0.61$$

EXPECTED VALUE OF PERFECT INFORMATION

EVWPI: "Expected Value With Perfect Information"

EVWOI: "Expected Value Without Information"

EVPI = EVWPI - EVWOI

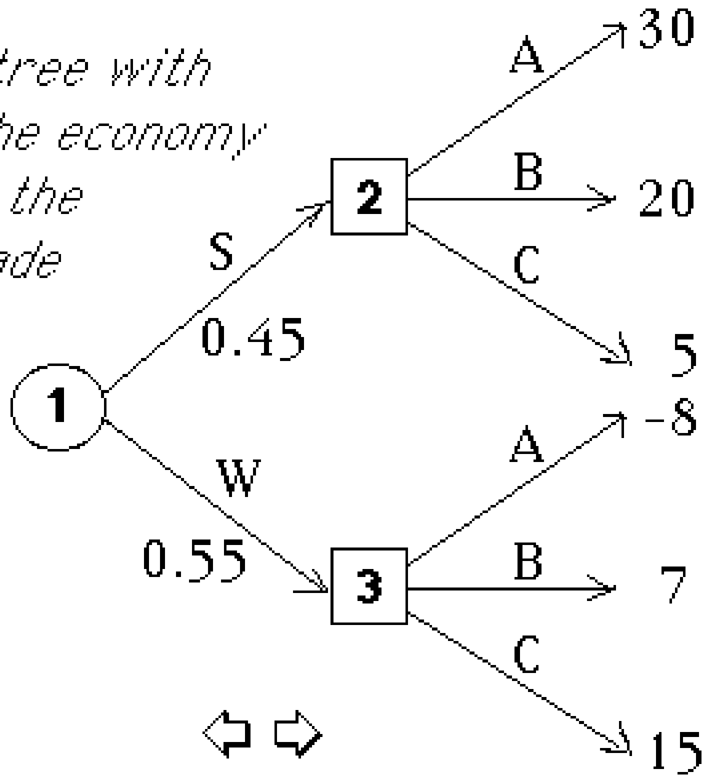
EXAMPLE PROTRAC decision problem

To calculate EVWPI ("Expected Value With Perfect Information"), we draw the decision tree in which the decision-maker has full knowledge of which state has occurred before the decision must be made.

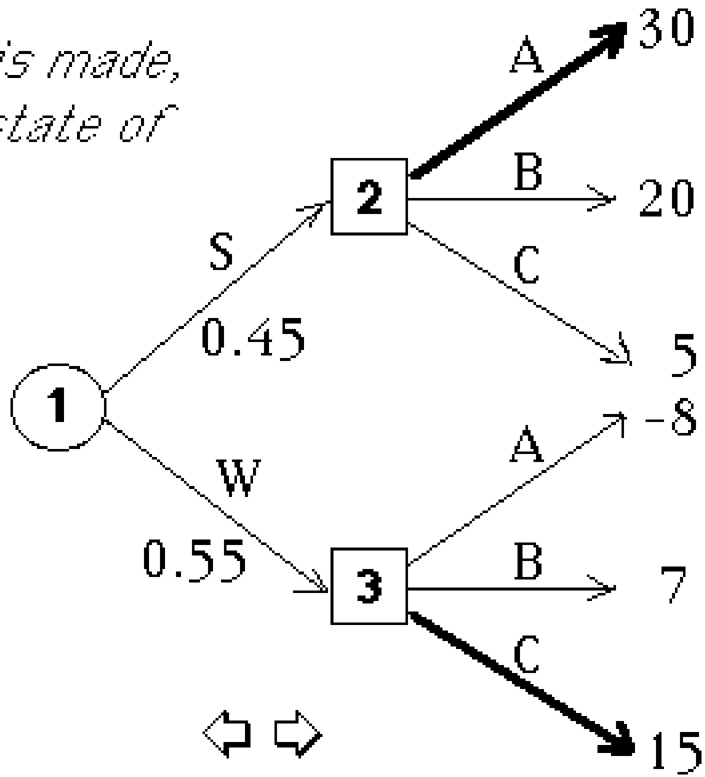
The Payoff Table

	State of "Nature"		
	S: strong	W: weak	
Decision	0.45	0.55	Probability
A	30	-8	
B	20	7	
C	5	15	

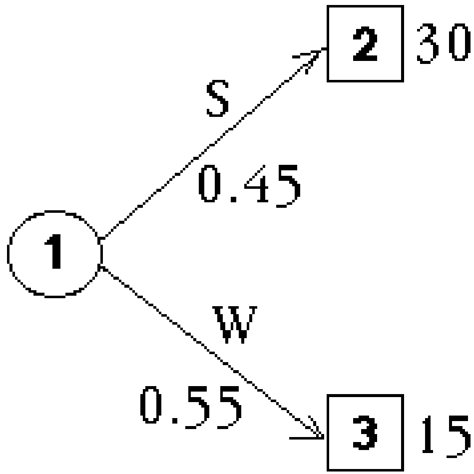
We draw the tree with the state of the economy known before the decision is made



The decision is made, knowing the state of the economy



Folding back:



$$0.45 \times 30 + 0.55 \times 15$$

(1) 15.65

EVWPI

EVPI = EVWPI - EVWOI

$$= 15.65 - 12.85$$

$$= 2.8$$

← *Expected Value of Perfect Information*

EXAMPLE

Farmer Jones must determine whether to plant

corn
or
soybeans

on a certain piece of land.

His "payoff" depends upon the weather conditions during the summer growing season:

- If he plants **corn** and the weather is **warm**, he earns $8000
- If he plants **corn** and the weather is **cold**, he earns $5000
- If he plants **soybeans** and the weather is **warm**, he earns $7000
- If he plants **soybeans** and the weather is **cold**, he earns $6500.

prior probabilities

In the past,

40% of all years have been **cold**,

and 60% have been **warm**.

Before planting, farmer Jones can pay $600 for an expert weather forecast.

If the year will actually be cold,

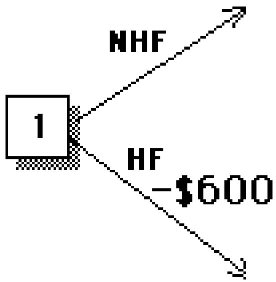
there is a 90% chance that the forecaster will be correct, i.e., predict a cold year.

If the year will actually be warm,

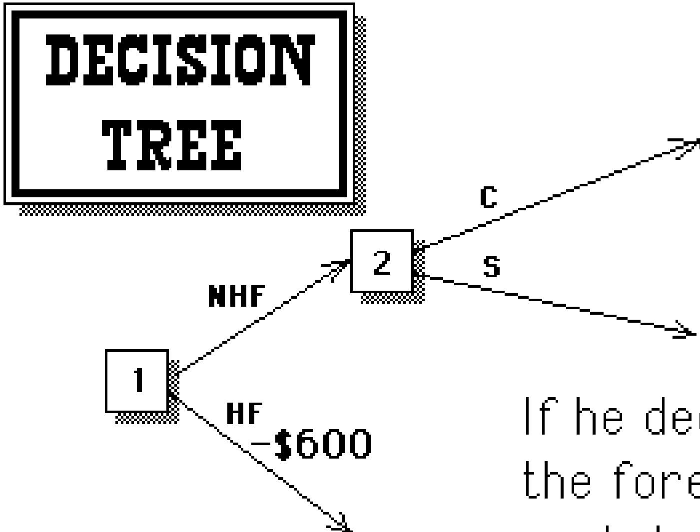
there is a 80% chance that the forecaster will be correct, i.e., will predict a warm year.

- CONSTRUCTING DECISION TREE
- FOLDING BACK TREE
- OPTIMAL DECISIONS
- EXPECTED VALUE OF FORECAST
- EXPECTED VALUE OF PERFECT INFORMATION

DECISION TREE



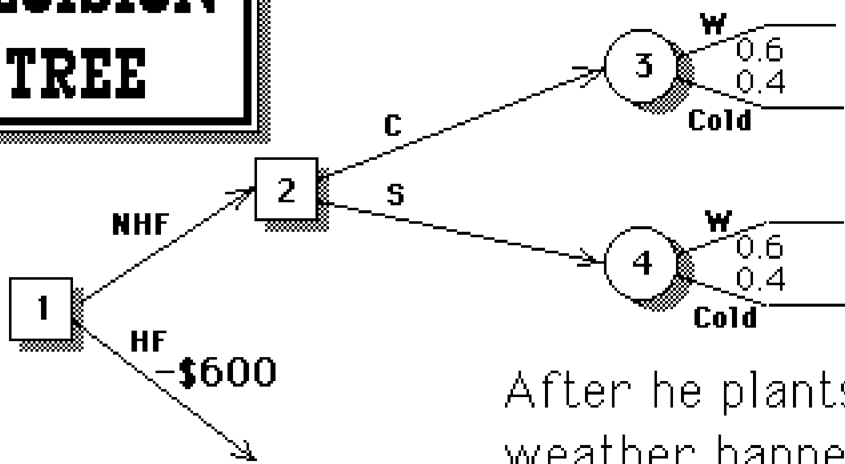
Jones must first decide whether to **H**ire **F**orecaster (**HF**), or **N**ot **H**ire **F**orecaster (**NHF**)



If he decides against hiring the forecaster, then he must next decide whether to plant

- Corn, or
- Soybeans

DECISION TREE



Prior Probabilities

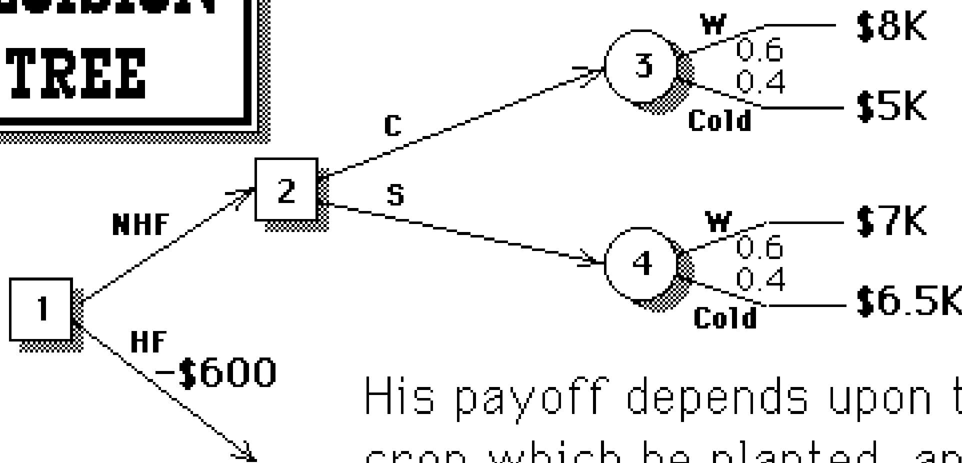
P{Warm}=60%

P{Cold} = 40%

After he plants his crop, the weather happens to be

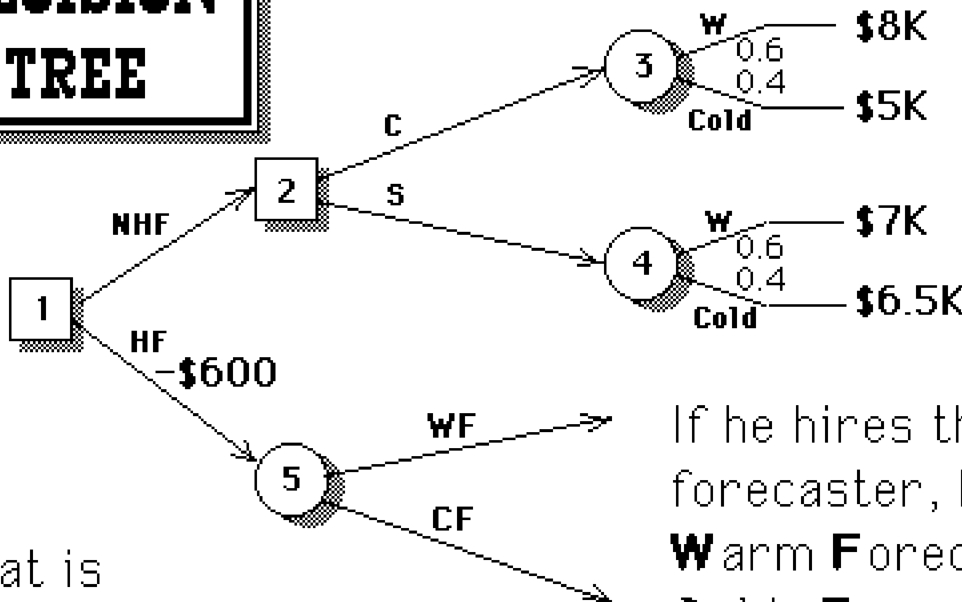
- **W**arm, or
- **C**old

DECISION TREE



His payoff depends upon the crop which he planted, and the weather conditions.

DECISION TREE



If he hires the forecaster, he is given **W**arm **F**orecast, or **C**old **F**orecast

What is P{WF} & P{CF}?

Condition the event "Warm Forecast" on the events "Warm weather" and "Cold weather":

P{Warm Forecast}

= P{Warm Forecast|Warm}P{Warm weather}
(correct in warm season)

+ P{Warm Forecast|Cold}P{Cold weather}
(error in cold season)

$$P\{WF\} = \underbrace{P\{WF|W\}}P\{W\} + \underbrace{P\{WF|C\}}P\{C\}$$

$$= 0.8 \times 0.6 + 0.1 \times 0.4$$

$$= 0.52$$

P{ Cold Forecast}

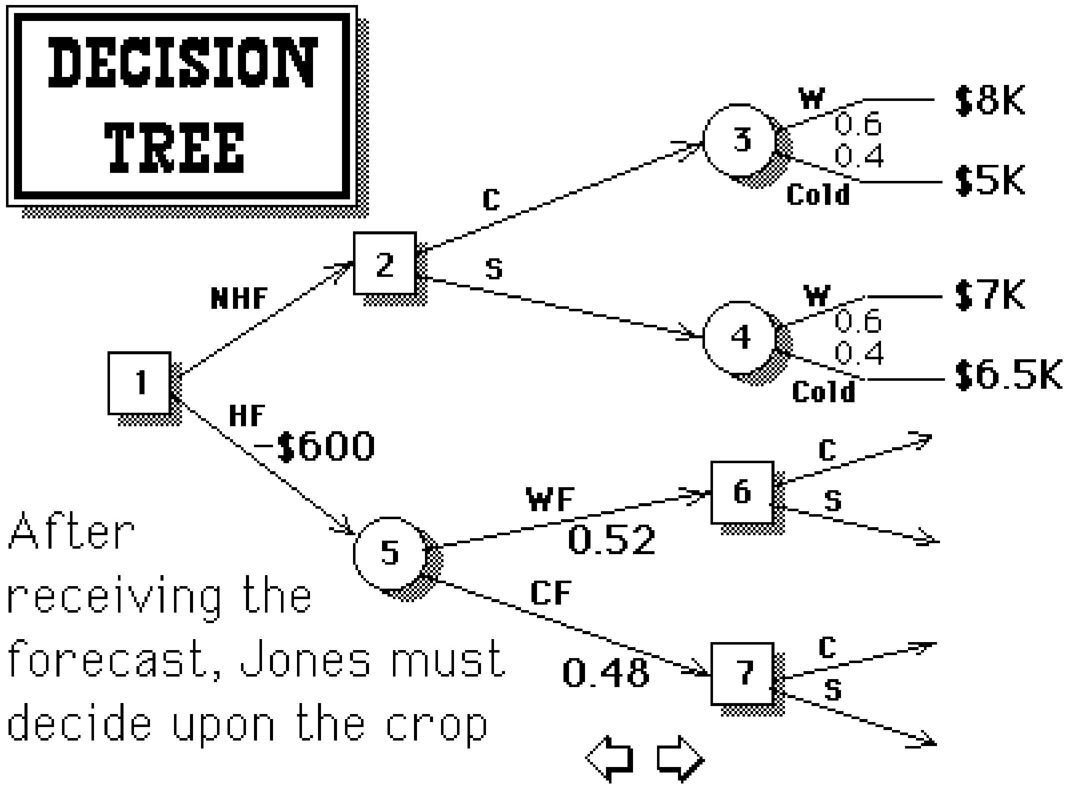
= P{Cold Forecast|Warm}P{Warm weather}
(error in warm season.)

+ P{Cold Forecast|Cold} P{Cold weather}
(correct in cold season.)

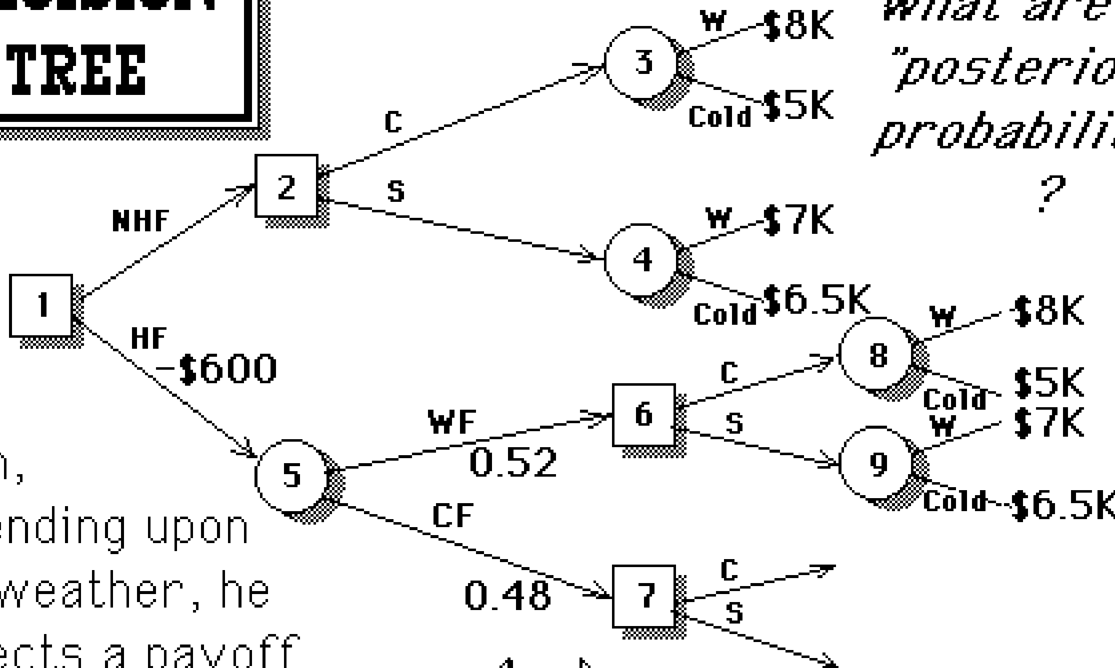
$$P\{CF\} = \underbrace{P\{CF|W\}}P\{W\} + \underbrace{P\{CF|C\}}P\{C\}$$

$$= \quad 0.2 \times 0.6 \quad + \quad 0.9 \times 0.4$$

$$= 0.48 = 1 - P\{WF\}$$


DECISION TREE
1
NHF
2
C
S
3
W
0.6
0.4
Cold
$8K
$5K
4
W
0.6
0.4
Cold
$7K
$6.5K
HF
-$600
5
WF
0.52
CF
0.48
6
C
S
7
C
S
After receiving the forecast, Jones must decide upon the crop

DECISION TREE



What are the "posterior probabilities ?

Then, depending upon the weather, he collects a payoff

Revised probabilities after receiving forecast

P{Warm weather | Warm Forecast}

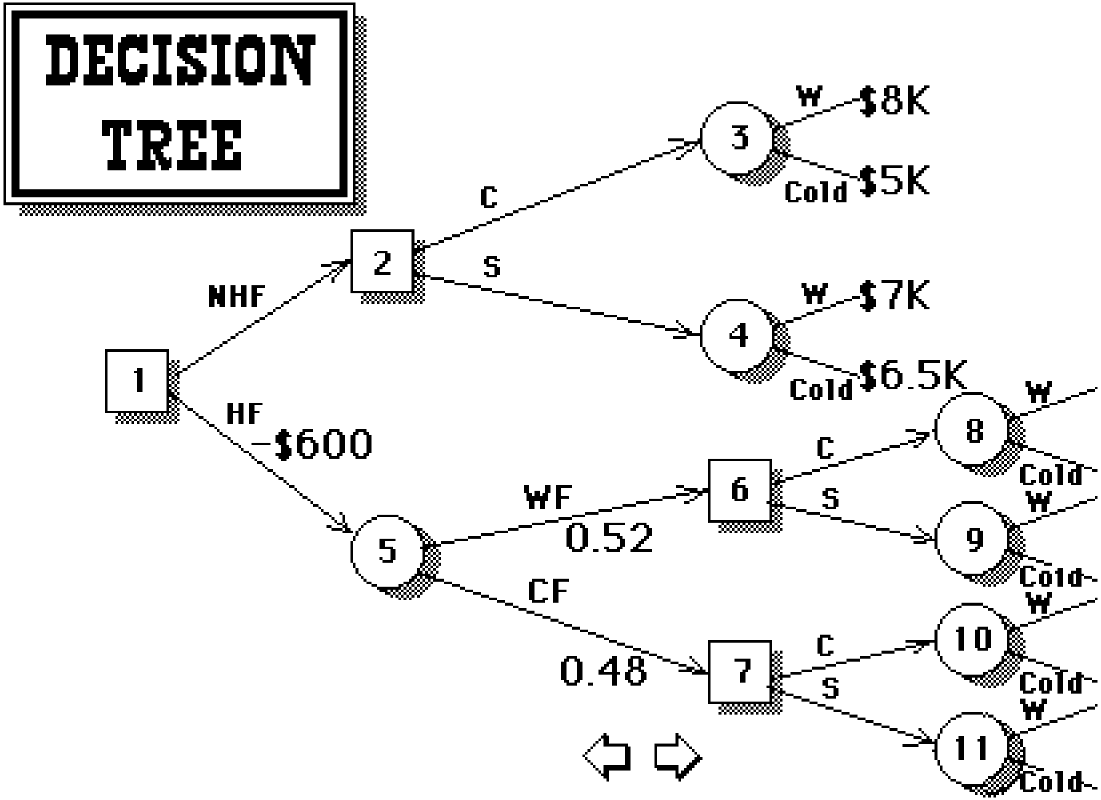
$$P\{W|WF\} = \frac{P\{WF|W\}\,P\{W\}}{P\{WF\}} = \frac{0.8 \times 0.6}{0.52} = 0.9231$$

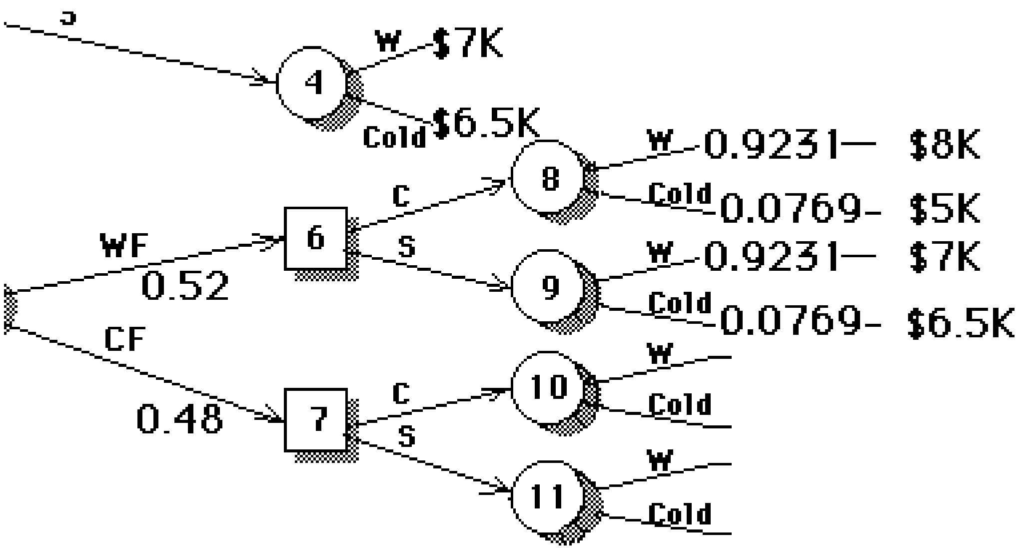
Bayes' Rule

posterior probabilities

P{Cold weather | Warm Forecast}

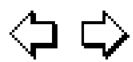
$$P\{C|WF\} = 1 - P\{W|WF\} = 0.0769$$


DECISION TREE
1
NHF
HF
-$600
2
C
S
3
W
$8K
Cold
$5K
4
W
$7K
Cold
$6.5K
5
WF
0.52
CF
0.48
6
C
S
7
C
S
8
W
Cold
9
W
Cold
10
W
Cold
11
W
Cold



$P\{W \mid WF\} = 0.9231$

$P\{C \mid WF\} = 0.0769$



Revised probabilities after receiving forecast

P{Cold | Cold Forecast} =

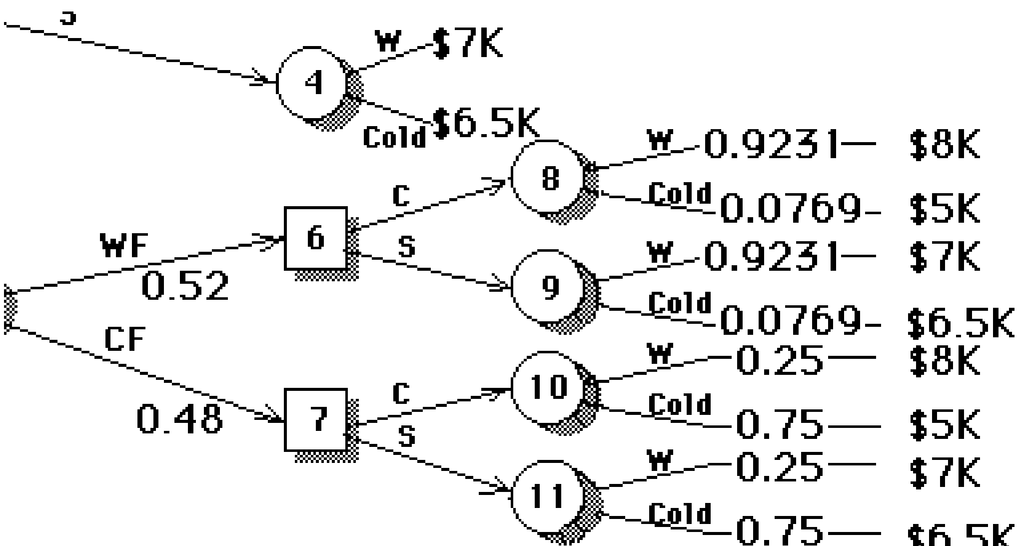
$$P\{C \mid CF\} = \frac{P\{CF \mid C\}\,P\{C\}}{P\{CF\}} = \frac{0.9 \times 0.4}{0.48} = 0.75$$

Bayes' Rule

posterior probabilities

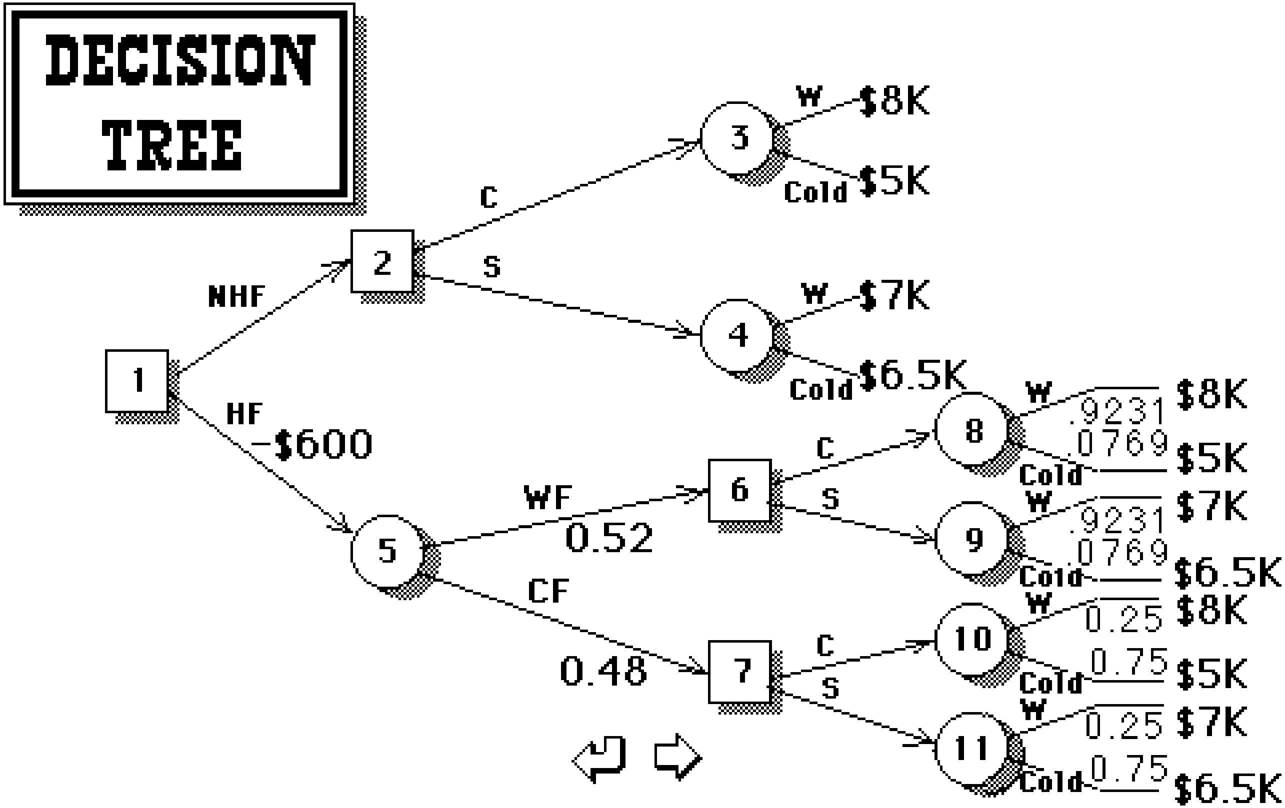
P{Warm | Cold Forecast}

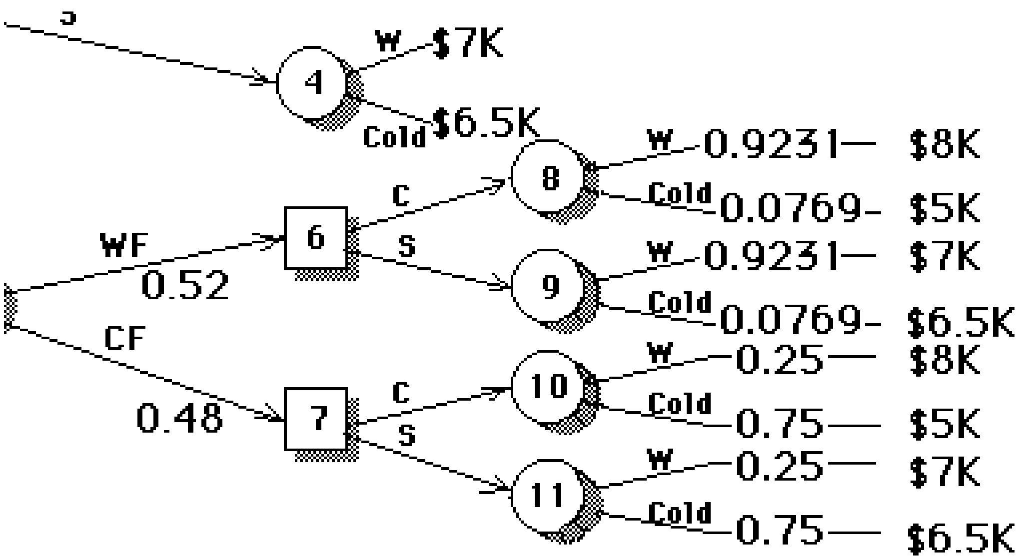
$$P\{W \mid CF\} = 1 - P\{C \mid CF\} = 1 - 0.75 = 0.25$$



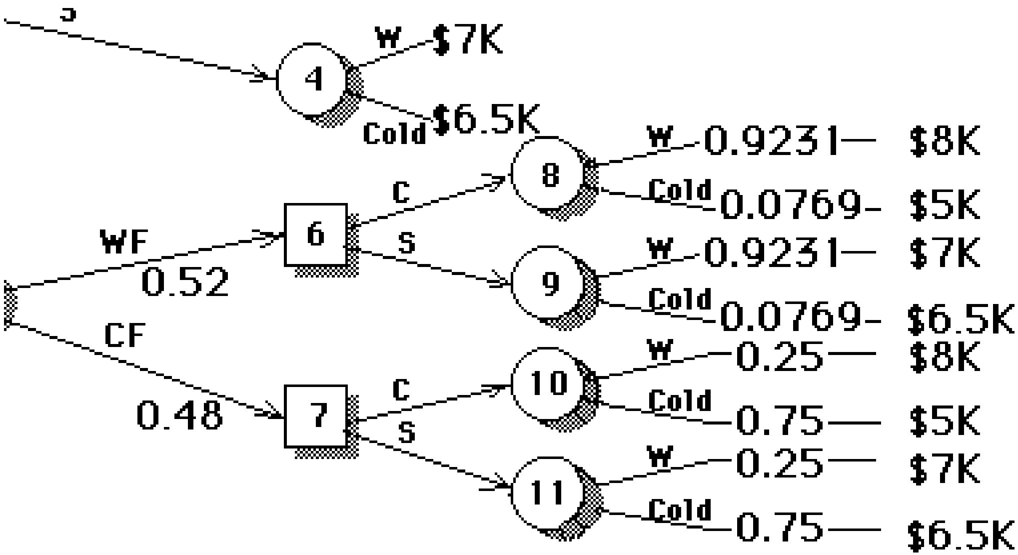
$P\{C|CF\} = 0.75$

$P\{W|CF\} = 0.25$


DECISION TREE
1
NHF
HF
-$600
2
C
S
3
W
$8K
Cold
$5K
4
W
$7K
Cold
$6.5K
5
WF
0.52
CF
0.48
6
C
S
7
C
S
8
W
.9231
$8K
.0769
Cold
$5K
9
W
.9231
$7K
.0769
Cold
$6.5K
10
W
0.25
$8K
0.75
Cold
$5K
11
W
0.25
$7K
0.75
Cold
$6.5K



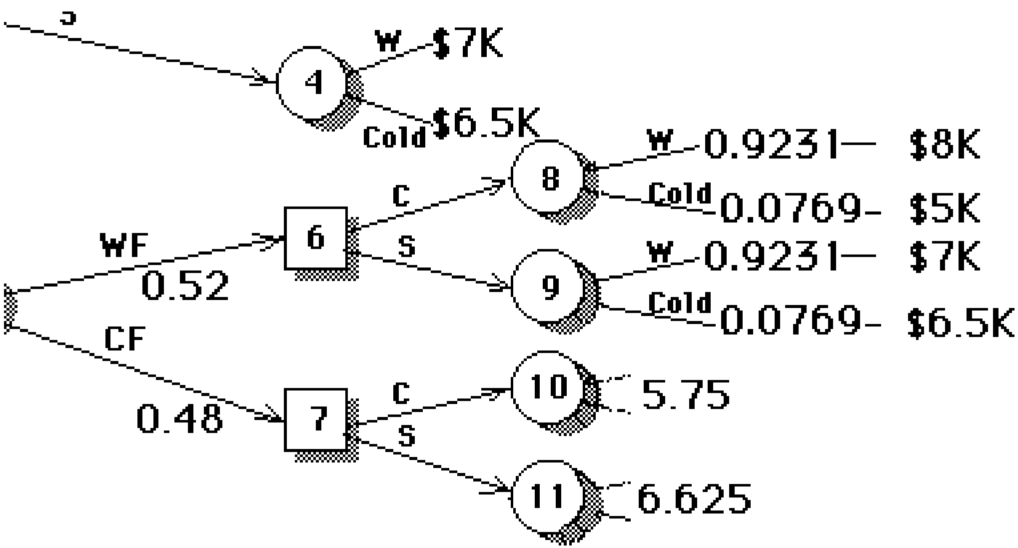
Now we begin "folding back"
the nodes of the tree...



Folding back nodes 10 & 11:

$$8(0.25) + 5(0.75) = 5.75$$

$$7(0.25)+6.5(0.75) = 6.625$$

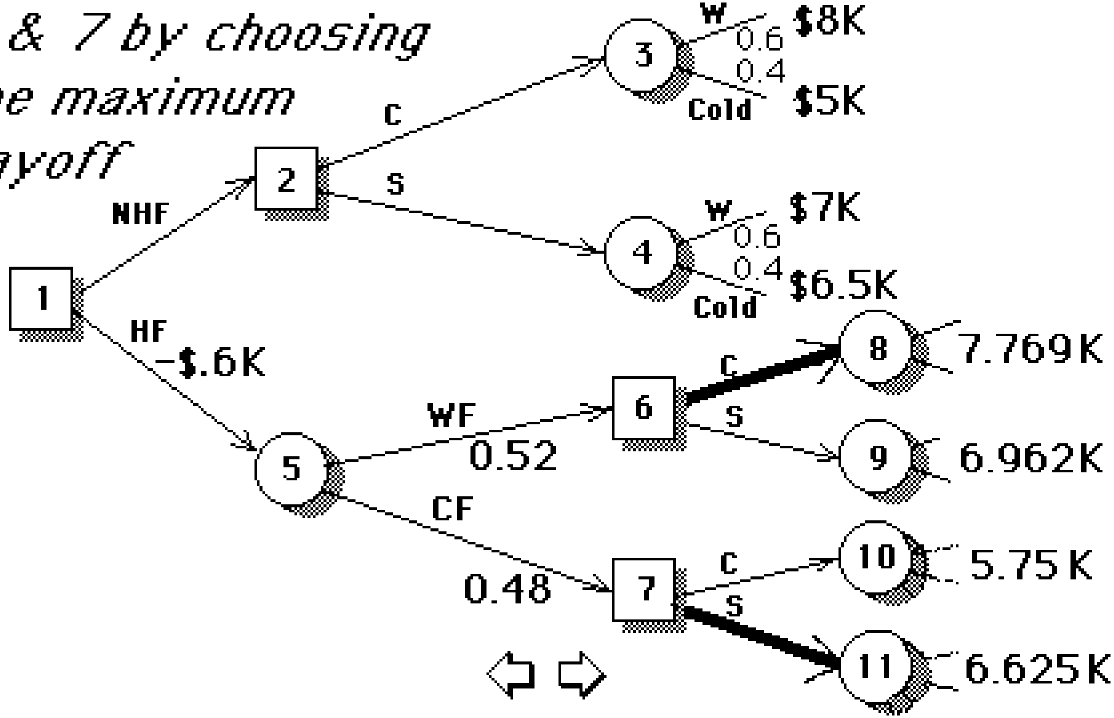


Folding back nodes 8 & 9:

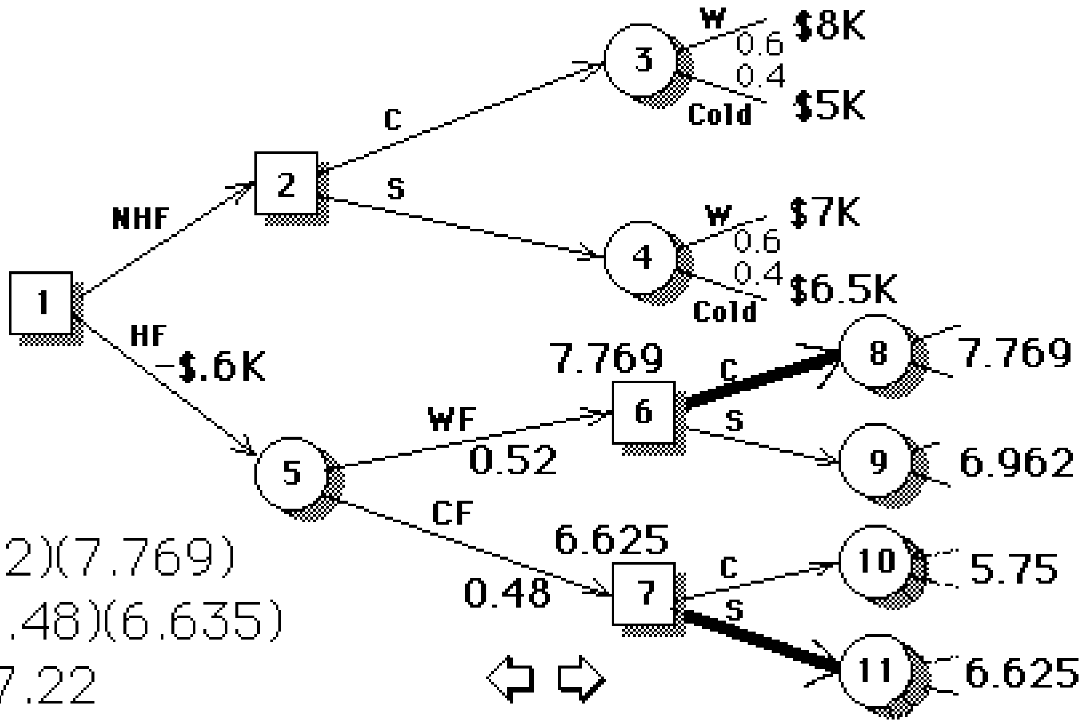
$$8(0.9231) + 5(0.0769) = 7.769$$

$$7(0.9231)+6.5(0.0769) = 6.962$$

Next we fold back nodes 6 & 7 by choosing the maximum payoff

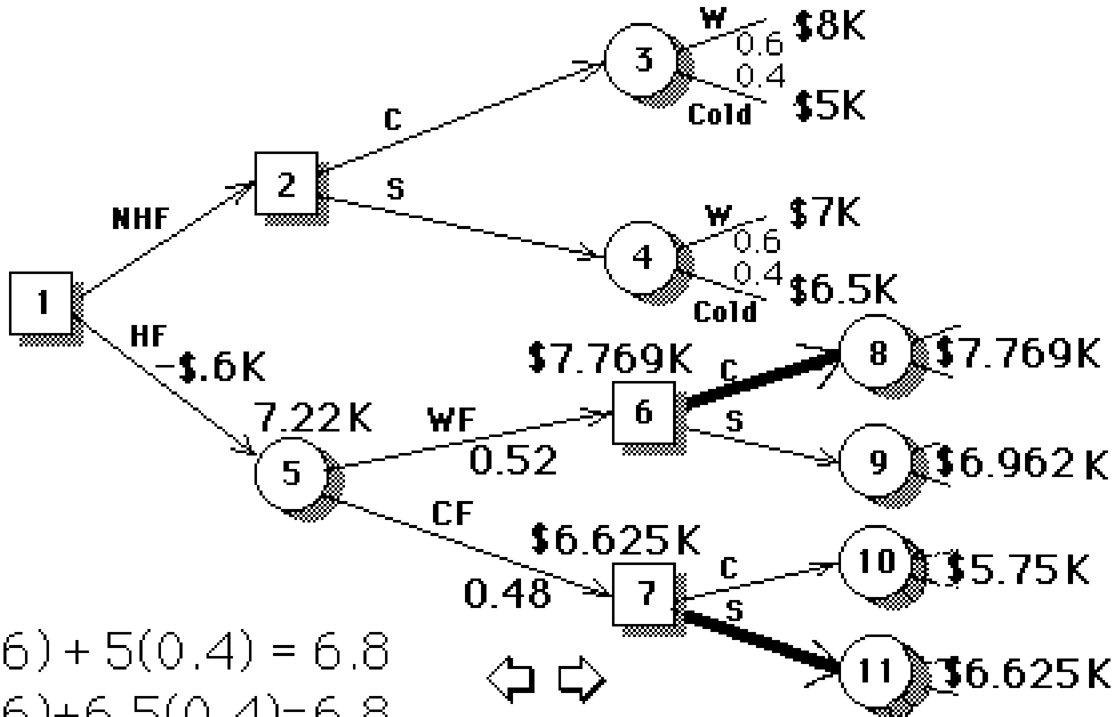


Next we fold back node 5:



$$(0.52)(7.769) + (0.48)(6.635) = 7.22$$

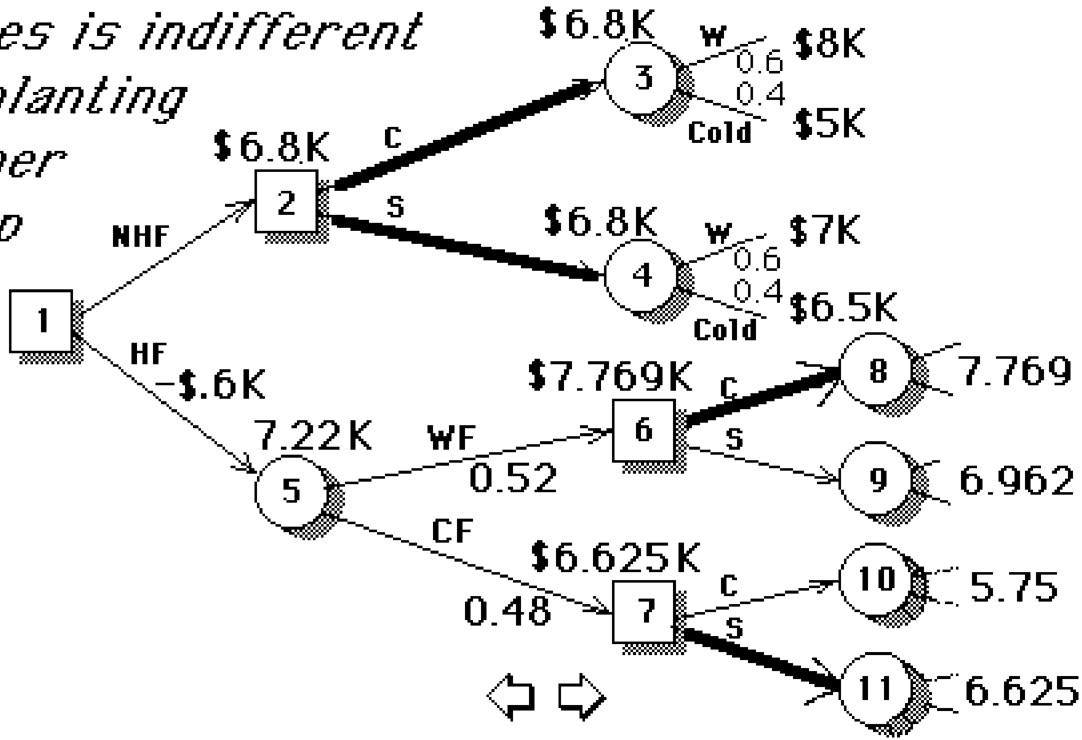
Next, fold back nodes 3 & 4:

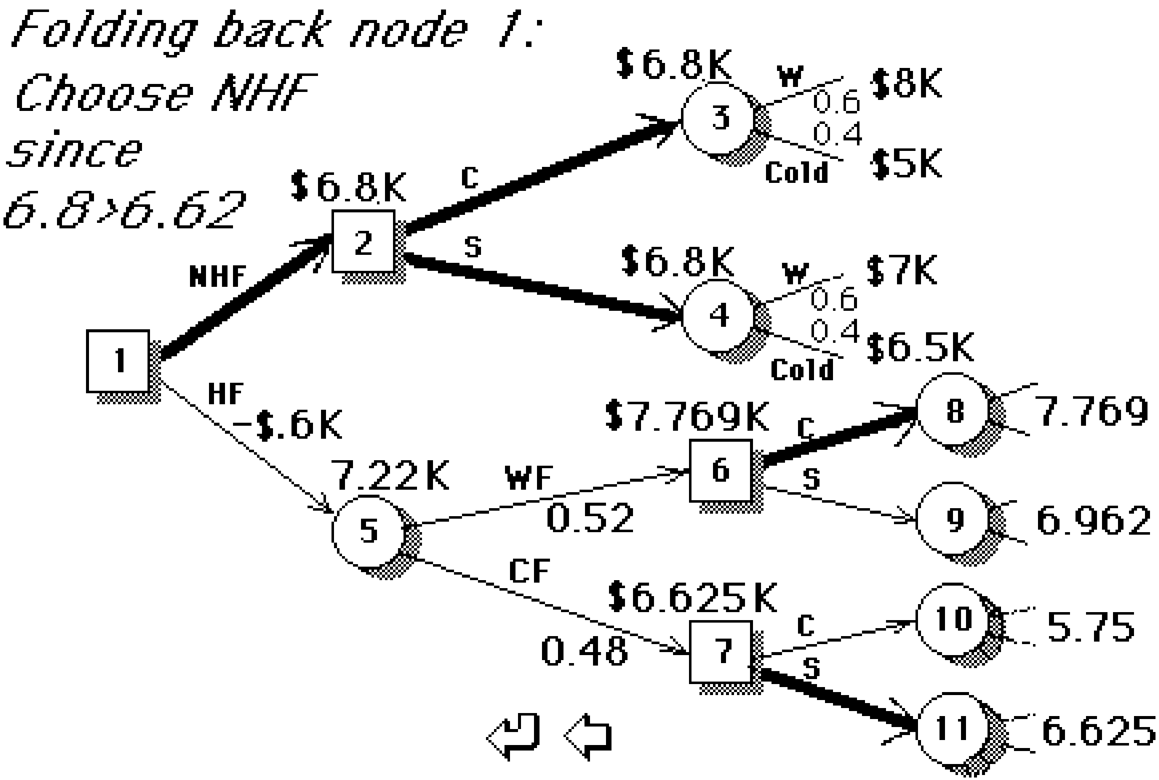


8(0.6) + 5(0.4) = 6.8

7(0.6)+6.5(0.4)=6.8

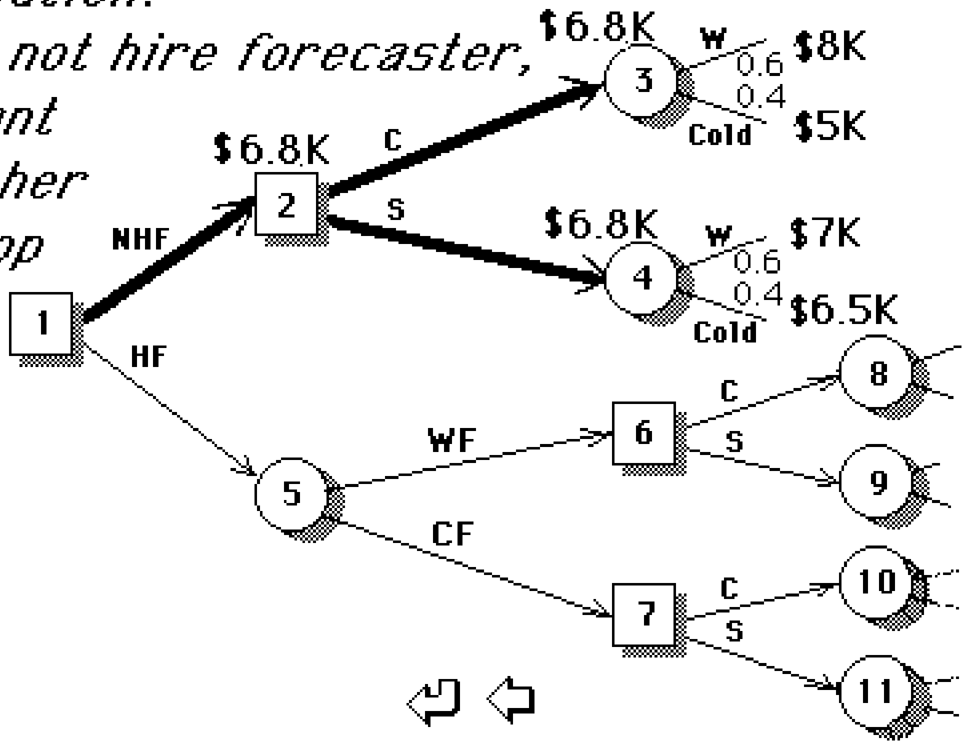
Folding back node 2: Jones is indifferent to planting either crop





Folding back node 1:
Choose NHF
since
6.8>6.62
$6.8K
$6.8K
$6.8K
$8K
$5K
$7K
$6.5K
W
0.6
0.4
Cold
C
S
NHF
HF
-$.6K
7.22K
$7.769K
$6.625K
WF
0.52
CF
0.48
7.769
6.962
5.75
6.625
1
2
3
4
5
6
7
8
9
10
11

Solution:
Do not hire forecaster, plant either crop



EVSI

What is the expected value of the forecast?

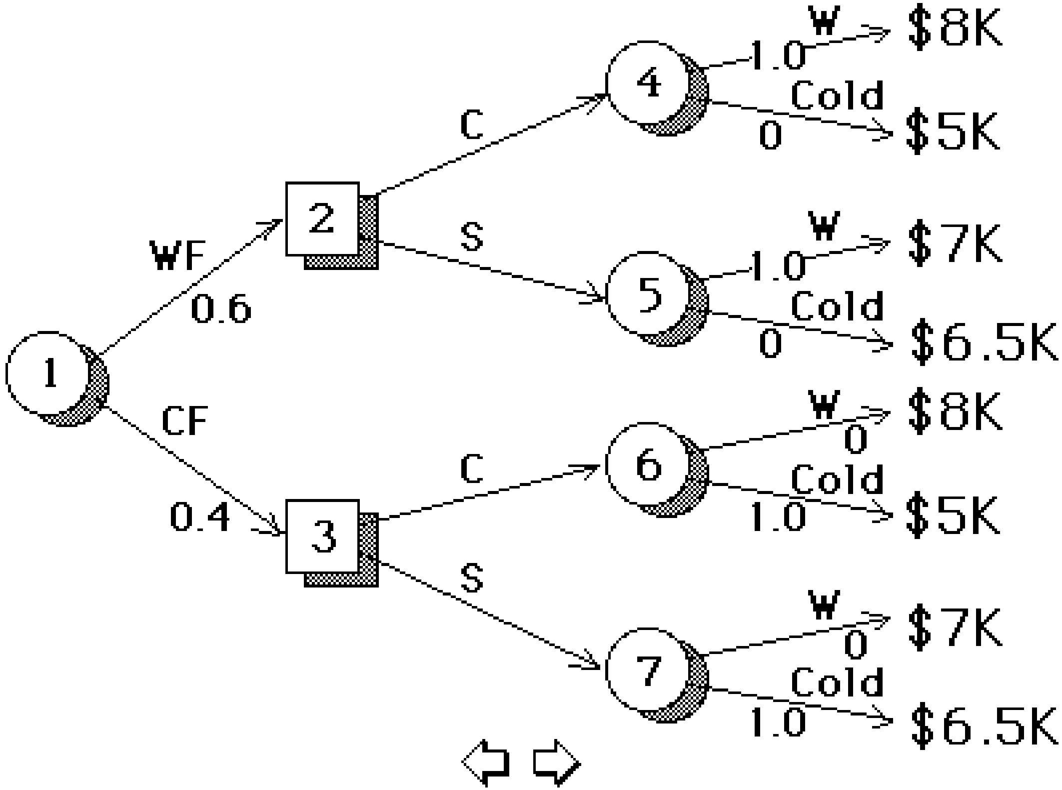
If the forecast were "free", Jones' expected payoff, using the forecast, would be $7.22K, or $420 more than his expected payoff without the forecast.

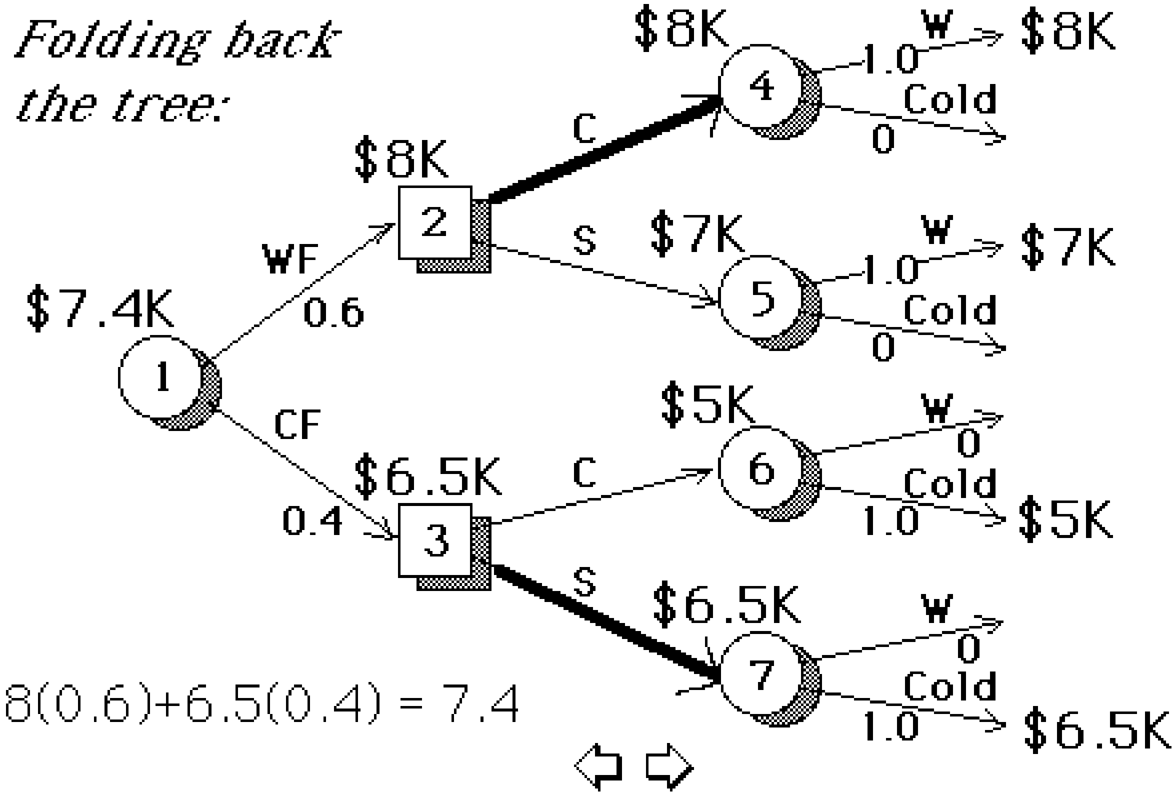
EVSI = $420

EVPI

What is the expected value of perfect information?

Imagine that Jones obtained a forecast which was 100% accurate


1
WF
0.6
CF
0.4
2
3
C
S
C
S
4
5
6
7
W
1.0
Cold
0
W
1.0
Cold
0
W
0
Cold
1.0
W
0
Cold
1.0
$8K
$5K
$7K
$6.5K
$8K
$5K
$7K
$6.5K


Folding back
the tree:
$7.4K
1
WF
0.6
CF
0.4
$8K
2
$6.5K
3
C
S
C
S
$8K
4
$7K
5
$5K
6
$6.5K
7
W
1.0
Cold
0
$8K
W
1.0
Cold
0
$7K
W
0
Cold
1.0
$5K
W
0
Cold
1.0
$6.5K
8(0.6)+6.5(0.4) = 7.4

$$EVPI = EVWPI - EVWOI$$

$$= \$7400 - \$6800 = \$600$$

Expected Value With Perfect Information

Expected Value Without Information

The NBS TV network earns an average of \$400K from a hit show, and loses an average of \$100K on a flop.

Of all shows reviewed by the network, 25% turn out to be hits and 75% flops.

For \$40K, a market research firm will have an audience view a prospective show and give its view about whether the show will be a hit or flop.

If a show is actually going to be a hit, there is a 90% chance that the market research firm will predict a hit; if the show is actually going to be a flop, there is an 80% chance that the firm will predict a flop.

What is the optimal strategy?

What is EVSI?

What is EVPI?