Discrete-Time Markov Chains

Models uncertainty in real-world systems that evolve dynamically in time.

Devised by the Russian mathematician A.A. Markov about 100 years ago to model the alternation of vowels and consonants in Pushkin's poetry.

Basic concepts

- states
- transition between states
- "Markovian" property: the future probabilistic behavior of the system depends *only* upon the present state of the system and *not* on any past history.

Definition:

The stochastic process $\{X_n, n=0,1,2,\ldots\}$ with state space I is a *discrete-time Markov chain* if, for each n=0,1,2,...

$$P\left\{X_{n+1}=j \mid X_0=i_0, X_1=i_1, \ldots X_n=i_n\right\}=P\left\{X_{n+1}=j \mid X_n=i_n\right\}=p_{i_n,j}^{n,n+1}$$

for all possible values of $i_0, i_1, \ldots i_{n+1}$.

We will consider only *stationary* (time-homogeneous) transition probabilities, that is, one-step transition probabilities

$$P\left\{X_{n+1}=j \mid X_n=i\right\}=p_{ij}$$

independent of the time parameter n.

Terminology & Notation:

- ♦ $p_{ij} = P\{X_{n+1} = j \mid X_n = i\}$: (stationary) transition probability that the system is next in state j *if* it is now in state i.
- ♦ $p_{ij}^{(n)} = P\{X_n = j \mid X_0 = i\}$: n-stage probability, i.e., probability that, at stage n, the system is in state j, given that it is initially in state i. Note that $p_{ij}^{(1)} \equiv p_{ij}$.
- ♦ $\pi_i = \lim_{n\to\infty} p_{ki}^{(n)}$, steadystate (equilibrium) distribution of the state of the system, independent of the initial state k

ⓘ *Note that the existence of the limiting steadystate distribution depends upon characteristics of the Markov chain, as described later!*

Terminology & Notation, *continued*

- N_{ij} = first-passage time (a random variable): number of stages required to reach state j for the ***first*** time, given that the process begins in state i
- $f_{ij}^{(n)} = P\left\{N_{ij} = n\right\}$: first-passage probability, the probability distribution of N_{ij}
- $f_{ij} \equiv \sum_{n=1}^{\infty} f_{ij}^{(n)}$: probability that a system which is initially in state i will eventually be found in state j.
- $m_{ij} = E\left\{N_{ij}\right\}$: mean first-passage time, the expected value of N_{ij}

Define the *n-step transition probabilities*

$$p_{ij}^{(n)} = P\{X_n = j \mid X_0 = i\}$$

That is, $p_{ij}^{(n)}$ is the probability that, if the system begins (at time n=0) in state i, it will be found in state j after n transitions.

Note that generally $p_{ij}^{(n)} \neq \left(p_{ij}\right)^n$! If, however, we form the matrix P with element p_{ij} in row *i* & column *j*, then we will find that $p_{ij}^{(n)}$ is the element in row *i* & column *j* of P^n, i.e., the n^th^ power of P. This is the essence of the *Chapman-Kolmogoroff equations*.

Chapman-Kolmogoroff Equations

For all stages n and m, and states i & j ∈ I,

$$p_{ij}^{(n+m)} = \sum_{k \in I} p_{ik}^{(n)} p_{kj}^{(m)}$$

Essentially, this simply states that $P^{n+m} = P^n P^m$.

ⓘ Example: (s,S) inventory replenishment system

State of system = inventory level, which is reviewed periodically, e.g., at end of business day

Random demands result in transition probability distributions

If inventory ≤ s, the inventory is replenished so as to raise the inventory level to S.

First-Passage Times

First-Passage Times

First-Passage Time N_{ij}: *(a random variable)* the number of stages required to reach state j for the first time, given that the system begins in state i.

That is,

$$N_{ij} = n \Leftrightarrow X_0 = i, X_k \neq j,\ \forall\ k < n, \text{ and } X_n = j$$

Denote by $f_{ij}^{(n)} = P\{N_{ij} = n\}$ the first-passage probabilities, i.e., the probability distribution of N_{ij}.

Note that $f_{ij}^{(1)} = p_{ij} \equiv p_{ij}^{(1)}$ but that, in general, $f_{ij}^{(n)} \leq p_{ij}^{(n)}$.

One may compute the probabilities $f_{ij}^{(n)}$ recursively.

Given that the initial state X_0 is i, express the probability that the system is in state j at the nth-step by conditioning upon the state k at which the system *first* reaches state j, using the "Law of Total Probability" which states that

$$p_{ij}^{(n)} = \sum_{k\le n} P\left\{X_n = j \mid \text{first visit to state j is in stage k}\right\} P\left\{\text{first visit to state j is in stage k}\right\}$$

$$= \sum_{k\le n} p_{jj}^{(n-k)} \times f_{ij}^{(k)} = \sum_{k<n} p_{jj}^{(n-k)} \times f_{ij}^{(k)} + p_{jj}^{(0)} f_{ij}^{(n)} = \sum_{k<n} p_{jj}^{(n-k)} \times f_{ij}^{(k)} + f_{ij}^{(n)}$$

Solve this equation for $f_{ij}^{(n)}$:

$$f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k<n} p_{jj}^{(n-k)} f_{ij}^{(k)} \text{ where } f_{ij}^{(1)} \equiv p_{ij}$$

Thus, the first-passage probabilities can be computed *recursively*, given sufficient powers of the matrix P.

ⓘCf. (s,S) inventory replenishment system

Mean First-Passage Times

Mean First-Passage Times

The expected value of the first-passage time is defined by the infinite sum:

$$m_{ij} \equiv E\left\{N_{ij}\right\} = \sum_{n=0}^{\infty} n f_{ij}^{(n)}$$

The ***mean first passage time*** can be computed approximately by including a large number of terms in the sum.

Fortunately there is another method which requires solving a finite *system of linear equations.*

The mean first passage times can more conveniently be computed by using the "Law of Total Expectation":

$$E\{N_{ij}\} = \sum_{k\in I} E\{N_{ij} \mid X_1 = k\} \times P\{X_1 = k\}$$

$$= E\{N_{ij} \mid X_1 = j\} P\{X_1 = j\} + \sum_{k\neq j} E\{N_{ij} \mid X_1 = k\} P\{X_1 = k\}$$

$$= 1 \times p_{ij} + \sum_{k\neq j} \left[1 + E\{N_{kj}\}\right] \times p_{ik}$$

That is,

$$E\{N_{ij}\} = p_{ij} + \sum_{k\neq j} p_{ik} + \sum_{k\neq j} E\{N_{kj}\} p_{ik}$$

$$m_{ij} = 1 + \sum_{k\neq j} p_{ik} m_{kj}$$

For fixed j, this gives us a system of *n* linear equations in n variables, m_{kj}, $k \in I$, where $n = |I|$.

ⓘ Cf. (s,S) inventory replenishment example.

Classification of States

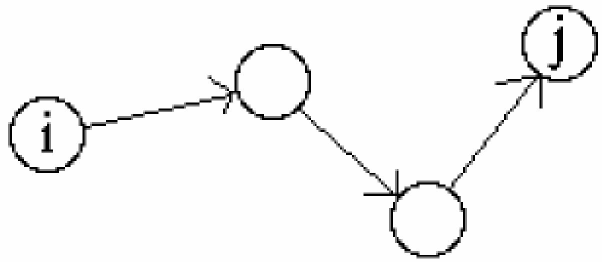
Classification of States

We will restrict our attention to Markov chains with a ***finite*** number of states.

Define $f_{ij} \equiv \sum_{n=1}^{\infty} f_{ij}^{(n)}$, the probability that the Markov chain will eventually be found in state j if it begins in state i.

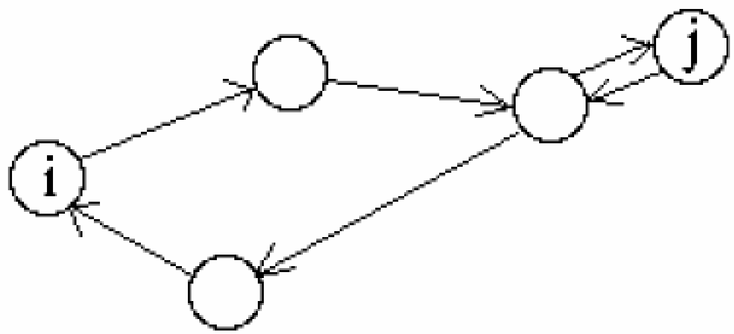
State i of a Markov chain may be classified as

- ♦ ***recurrent*** if $f_{ii} = 1$, i.e., the system is certain to return to state i if it begins in state i
- ♦ ***transient*** if $f_{ii} < 1$, i.e., there is positive probability that the system, beginning in state i, fails to return to this state.



State j is ***reachable*** from state i

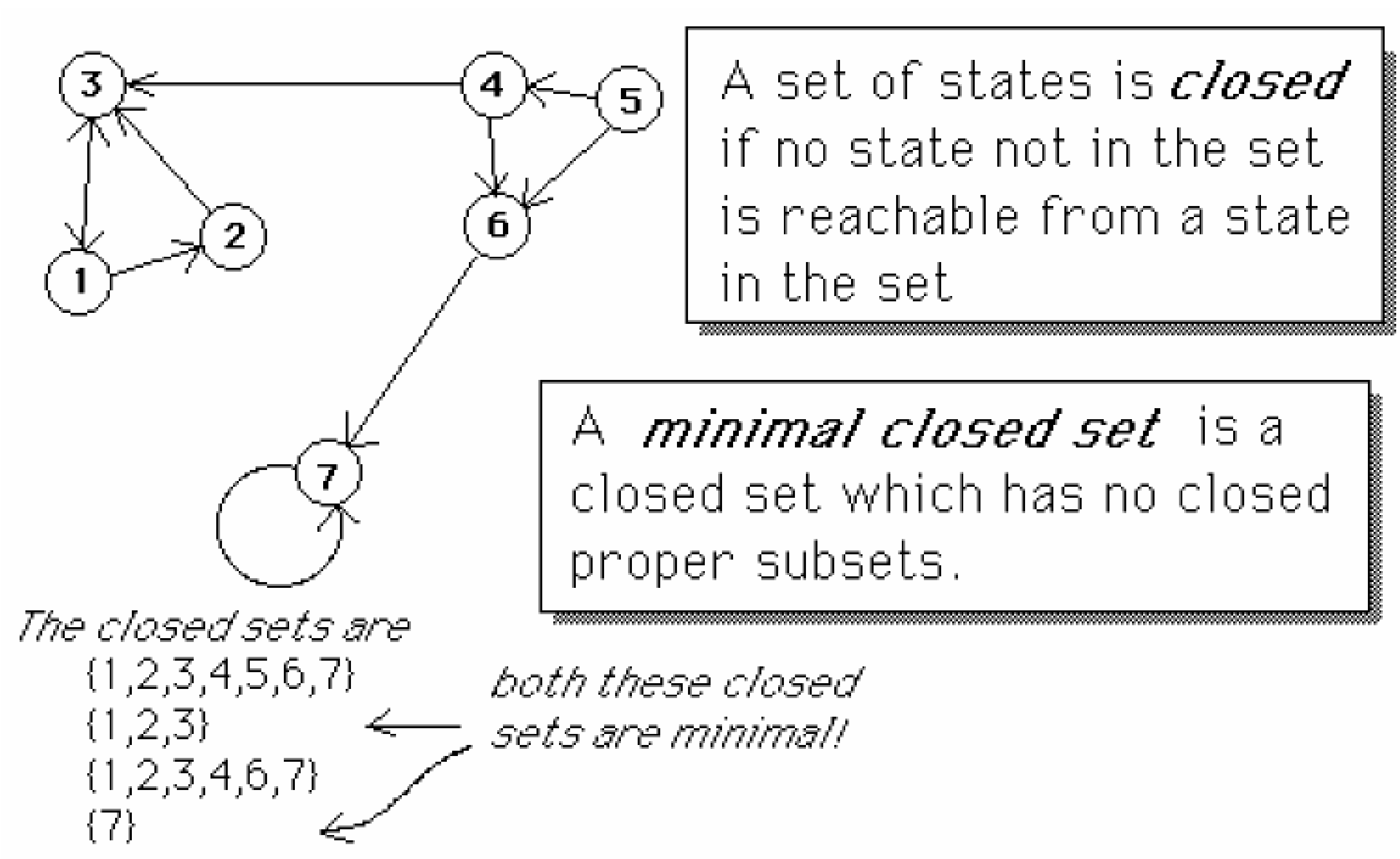
$i \longrightarrow j$



States i & j ***communicate***

$i \longleftrightarrow j$

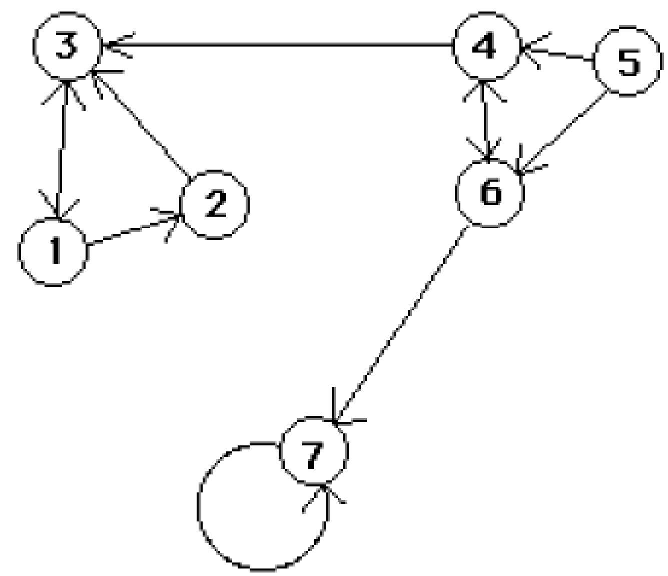
If state i is recurrent, and states i & j communicate, then state j is recurrent.



Note: "minimal" does not refer to the cardinality of the set…. two minimal closed sets may have different cardinality!

A minimal closed set is also said to be ***irreducible***.

The concept of minimal closed set gives us another characterization of recurrent states:



States 1,2,3, & 7 are recurrent.

In a Markov chain with finitely many states, a member of a minimal closed set is ***recurrent*** and other states are ***transient***

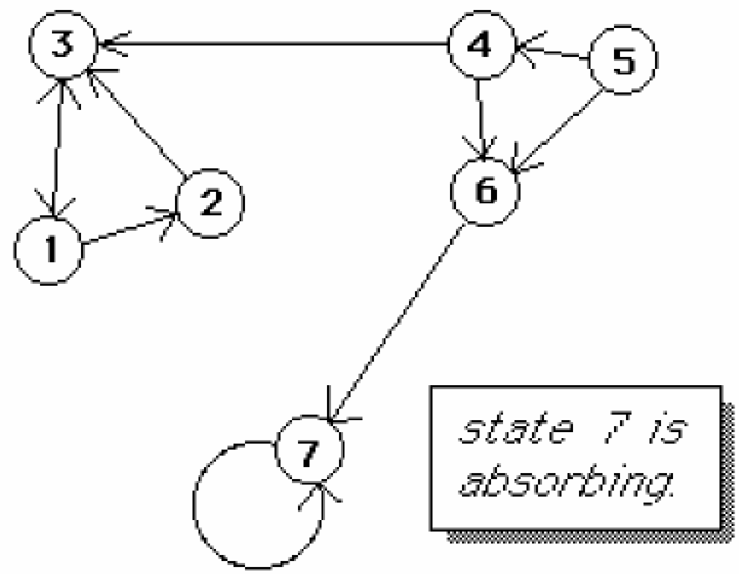
Absorbing States

A state which forms a closed set, i.e., which cannot reach another state, is said to be ***absorbing***.

If state j is absorbing, then

$$p_{jj} = p_{jj}^{(n)} = 1$$

for all n=1, 2, ...



If a Markov chain has absorbing states, the states might be reordered so that the transition probability matrix P is of the form

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

where the size of the identity matrix I is the number of absorbing states.

When there are more than one absorbing state, a question which is frequently of interest is

"If the system begins in a transient state i, what is the probability that the system eventually reaches (and hence is absorbed) into state j?"

Absorption Probabilities

When there are r>0 absorbing states, the powers of the transition probability matrix P will be of the form

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}, P^2 = \begin{bmatrix} Q^2 & R+QR \\ 0 & I \end{bmatrix}, P^3 = \begin{bmatrix} Q^3 & R+QR+Q^2R \\ 0 & I \end{bmatrix},$$

$$\cdots\cdots P^n = \begin{bmatrix} Q^n & \left(R+QR+Q^2R+\cdots+Q^{n-1}R\right) \\ 0 & I \end{bmatrix} = \begin{bmatrix} Q^n & \left(I+Q+Q^2+\cdots+Q^{n-1}\right)R \\ 0 & I \end{bmatrix}$$

But the series

$$(I-Q)\left(I+Q+Q^2+Q^3+\cdots\right) = I-Q+Q-Q^2+Q^2-Q^3+Q^3-\cdots = I$$

That is, the infinite series is the inverse of the difference (I-Q):

$$(I-Q)^{-1} = I+Q+Q^2+Q^3+\cdots$$

Define the limit

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty}\begin{bmatrix} Q^n & \left(\sum_{k=1}^{n-1} Q^k\right)R \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & ER \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$$

where $E = \sum_{k=0}^{\infty} Q^k$.

That is, the square matrix Q^n consists of the n-step transition probabilities from a transient state to another transient state, and the (n-r]×r matrix A=ER consists of the probabilities of absorption into an absorbing state, beginning from a transient state.

Let states i & j both be transient, and define

e_{ij} = expected # of visits to state j, given that the system begins in state i (counting initial visit if i=j)

$$e_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

and the $r \times r$ matrix:

$$E = \sum_{n=0}^{\infty} Q^n = (I - Q)^{-1}$$

since $(I - Q)\left(I + Q + Q^2 + \dots\right) = I + Q - Q + Q^2 - Q^2 + \dots = I$

See examples:

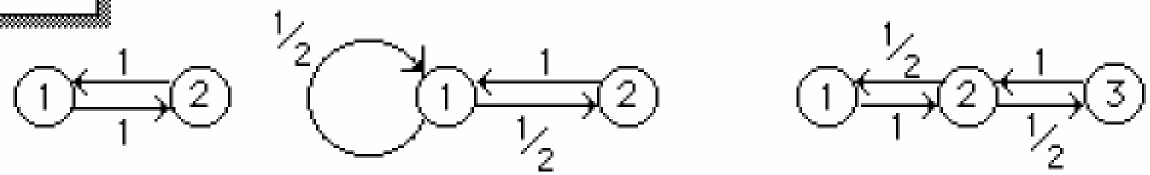
- ♦ Markov chain analysis of a multistage manufacturing system with inspection and reworking. What fraction of the parts which begin the process are eventually scrapped?
- ♦ "Passing the Buck"-- what fraction of the operating expenses of a service facility should be allocated to the production units?

Periodicity

The *period* $d(i)$ of state i is the greatest common divisor of all the integers $n \geq 1$ for which

$$p_{ii}^{(n)} > 0$$

Examples



d(1)=d(2)=2 d(1)=d(2)=1 d(1)=d(2)=d(3)=2

If $i \leftrightarrow j$, then $d(i)=d(j)$.

A Markov chain with $d(i)=1$ for all i is called *aperiodic*

Conditions for Existence of Steadystate Distribution

The ***Unichain Assumption*** concerning a finite-state Markov chain:

The Markov chain has only one minimal closed set of recurrent states and a (possibly empty) set of transient states.

Theorem

Let $\{X_n\}$ be a finite-state aperiodic Markov chain satisfying the Unichain Assumption. Then there exists a probability distribution π such that

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \quad \text{for all } j=1,2,\ldots n$$

Characterization of the Steadystate Distribution

Consider a finite-state aperiodic Markov chain satisfying the Unichain Assumption. Then the limiting distribution π in the previous theorem satisfies the *equilibrium conditions*

$$\pi_j = \sum_{k=1}^{n} \pi_k p_{kj} \quad \text{for each j=1,2,...n}$$

or, in matrix representation,

$$\pi = \pi P$$

The vector **x**=0 satisfies these equilibrium conditions; furthermore, if x is a solution, then any scalar multiple of x also satisfies the equations. However, adding the *normalizing* equation

$$\sum_{j=1}^{n} \pi_j = 1$$

uniquely determines the limiting distribution.

Computing the Steadystate Distribution

The steadystate equations may be found by solving the system of linear equations

$$\begin{cases} \pi = \pi P \\ \sum_i \pi_i = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} (I-P)^T \pi = 0 \\ \sum_i \pi_i = 1 \end{cases}$$

Notes:

- The coefficients in each row of the system are obtained from the *columns* of P!
- The equations $\pi = \pi P$ are not full row rank, and include one *redundant* equation-- any one of the equations may be discarded.
- The system may be solved by *Gauss elimination*; if extremely large, *Gauss-Seidel* (successive overrelaxation, **SOR**) methods may be advantageous.

Example

Consider the Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.5 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

The system of equations determining the steadystate distribution is

$$\begin{cases} \pi = \pi P \\ \sum_i \pi_i = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = 0.4\pi_1 + 0.3\pi_2 + 0.6\pi_3 \\ \pi_2 = 0.5\pi_1 + 0.2\pi_2 + 0.2\pi_3 \\ \pi_3 = 0.1\pi_1 + 0.5\pi_2 + 0.2\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \begin{cases} 0.6\pi_1 - 0.3\pi_2 - 0.6\pi_3 = 0 \\ -0.5\pi_1 + 0.8\pi_2 - 0.2\pi_3 = 0 \\ -0.1\pi_1 - 0.5\pi_2 + 0.8\pi_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Discarding (arbitrarily) the 1st equation and applying ***Gauss elimination***:

$$\begin{bmatrix} -0.5 & 0.8 & -0.2 & 0 \\ -0.1 & -0.5 & 0.8 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1.6 & 0.4 & 0 \\ 0 & -0.66 & 0.84 & 0 \\ 0 & 2.6 & 0.6 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1.6 & 0.4 & 0 \\ 0 & 1 & {}^{-}1.27273 & 0 \\ 0 & 0 & 3.90909 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1.6 & 0.4 & 0 \\ 0 & 1 & {}^{-}1.27273 & 0 \\ 0 & 0 & 1 & 0.25581 \end{bmatrix}$$

Then ***back-substitution*** yields the solution:

$$\begin{cases} \pi_3 = 0.25581 \\ \pi_2 = 1.27273\pi_2 = 0.32558 \\ \pi_1 = 1.6\pi_2 - 0.4\pi_3 = 0.41861 \end{cases}$$