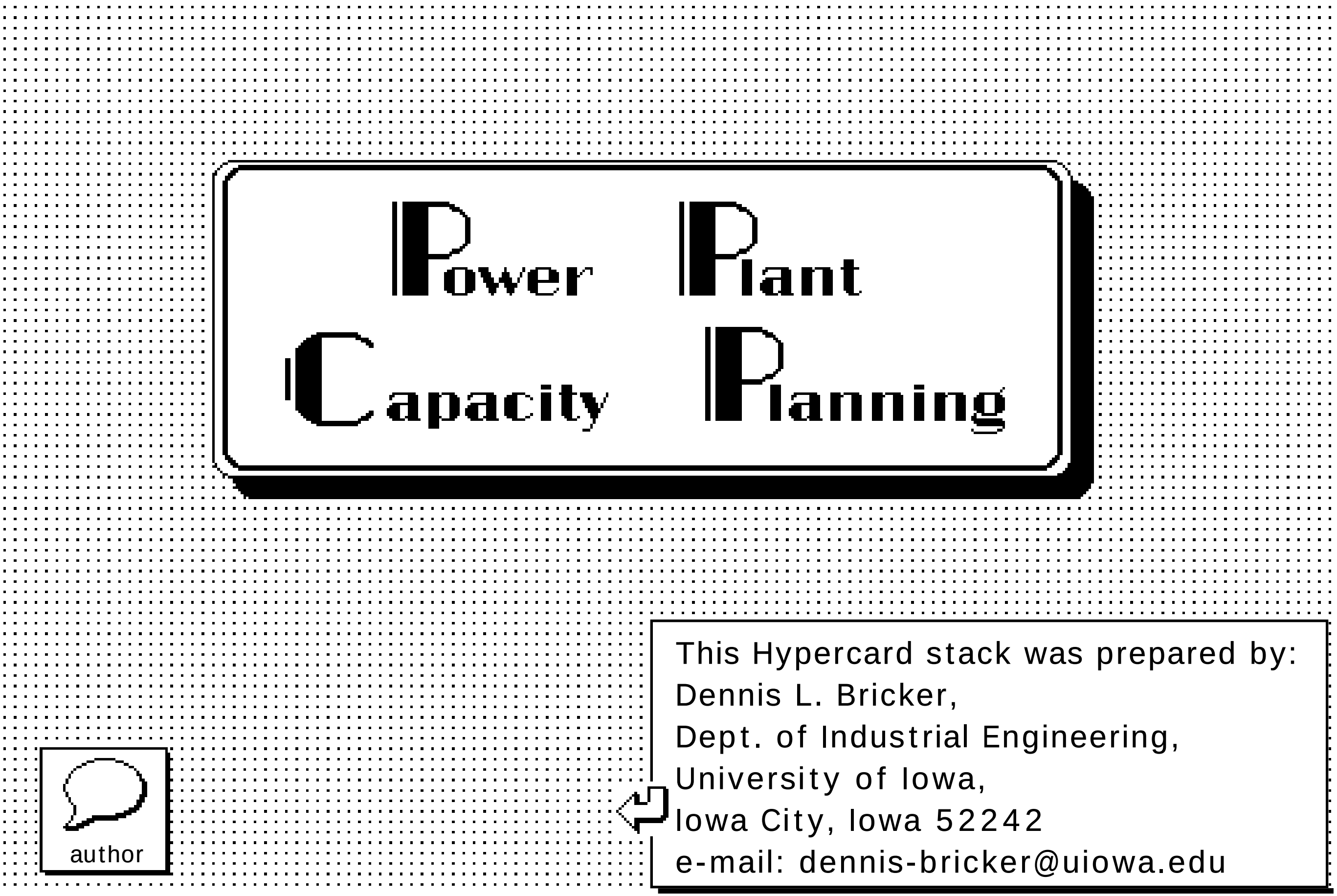

Power Plant Capacity Planning

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A power company is doing long-range planning, and has projected the following needs for additional power plants:

year	1	2	3	4	5	6
#plants req'd	1	2	4	6	7	8
cost/plant ($million)	5.4	5.6	5.8	5.7	5.5	5.2

cumulative!

There is a fixed cost of $1.5 million if any plants are added in a year

At most 3 plants can be added in one year

When should each of the 8 needed plants be added?

($ might be saved by adding some plants in advance of the year in which they're needed, thereby avoiding the $1.5million fixed cost each year.)

DP Model

Stage: n = # of years remaining in the planning period

State: S_n = total # of plants which have been added

Decision: X_n = # of plants to be added in the current year

(Note that the stages are being numbered in reverse chronological order! That is, stage 6 is the first year in the planning period, etc.)

Immediate cost of stage n

assures that sufficient plants have been added in previous year.

$$g_n(S_n,X_n) = \begin{cases} +\infty & \text{if } S_n < R_{n+1} \\ 0 & \text{if } X_n = 0 \quad \& \; S_n \ge R_n \\ 1.5+C_nX_n & \text{if } X_n=1,2, \text{ or } 3 \quad \& \; S_n \ge R_n \end{cases}$$

fixed cost

cost per plant

Transition Function

If the decision X_n is made, when the current state is S_n, the resulting state will be

$$\boxed{\underbrace{T(S_n,X_n)}_{\text{cumulative number of plants at beginning of next year}} = \underbrace{S_n}_{\text{current number of plants}} + \underbrace{X_n}_{\text{number of plants added}}}$$

Objective

- ☞ **Minimizing Total Cost**
- ☞ **Minimizing *Present Value* of Total Cost**

Definition of Optimal Value

$f_n(S_n)$ = minimum total cost of the current & remaining stages (n, n−1, ..., 0) if stage n is entered in state S_n

$$f_n(S_n) = \begin{cases} \displaystyle\min_{(R_n - S_n)^+ \le X_n \le 3} \left\{ g_n(S_n, X_n) + f_{n-1}(S_n + X_n) \right\} & \text{if } n \ge 1 \\ 0 & \text{if } n=0 \ \& \ S_o \ge R_1 \\ +\infty & \text{if } n=0 \ \& \ S_o < R_1 \end{cases}$$

$$\boxed{z^+ \equiv \max\{0, z\}}$$

APL code

Power Plant Capacity Planning

```
z←F N;t;S;G
⍝
⍝              Optimal Value Function
⍝          Power Plant Capacity Planning problem
⍝
→End IF N=0

⍝       Cost of adding plants in current year
G←((ρs)ρ0)  ∘.+  (x>0)×1.5+COST[N]×x

⍝     Cumulative # of plants added at beginning of next yr.
S← s∘.+x

⍝       Recursive definition of optimal value
z← MIN G + (F N-1)[TRANSITION S+BIG×(S<RQMT[N])]
→0

End:z←((ρs)ρ0),BIG
```

Stage 1

s \ x:	0	1
7	999999.99	6.70
8	0.00	999999.99

Cost per plant = C_1 = 5.2

Req'd # of plants = R_1 = 8

R_2 = 7

999999.99 represents infinity (infeasible decision).

Stage 1

s \ x:	0	1
7	999999.99	6.70
8	0.00	999999.99

State	Optimal Values	Optimal Decisions	Resulting State
7	6.70	1	8
8	0.00	0	8
	$f_1(S_1)$	X_1^*	S_0^*

Stage 2

from computations just completed:

S_1	$f_1(S_1)$
7	6.70
8	0.00

$$f_2(S_2) = \underset{(R_2-S_2)^+ \le X_2 \le 3}{\text{minimum}} \; g_2(S_2,X_2) + f_1(S_2+X_2)$$

Cost per plant = C_2 = 5.5

Req'd number of plants = R_2 = 7

s \ x:	0	1	2
6	999999.99	13.70	12.50
7	6.70	7.00	999999.99
8	0.00	999999.99	999999.99

$1.5 + 5.5\times1 + f_1(6+1)$

Stage 2

s \ x:	0	1	2
6	999999.99	13.70	12.50
7	6.70	7.00	999999.99
8	0.00	999999.99	999999.99

State	Optimal Values	Optimal Decisions	Resulting State
6	12.50	2	8
7	6.70	0	7
8	0.00	0	8

$\underbrace{\text{Optimal Values}}_{f_2(S_2)}$ $\underbrace{\text{Optimal Decisions}}_{X_2^*}$ $\underbrace{\text{Resulting State}}_{S_1^*}$

Stage 3

Cost per plant = C_3 = 5.7

Req'd number of plants = R_3 = 6

S_2	$f_2(S_2)$
6	12.50
7	6.70
8	0.00

$$f_3(S_3) = \underset{(R_3-S_3)^+ \le X_3 \le 3}{\textbf{minimum}} \; g_3(S_3,X_3) + f_2(S_3+X_3)$$

S_3 \ X_3	0	1	2	3
4	999999.99	999999.99	25.40	25.30
5	999999.99	19.70	19.60	18.60
6	12.50	13.90	12.90	999999.99
7	6.70	7.20	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

$1.5+5.7\times2 + f_2(5+2)$

s \ X:	0	1	2	3
4	999999.99	999999.99	25.40	25.30
5	999999.99	19.70	19.60	18.60
6	12.50	13.90	12.90	999999.99
7	6.70	7.20	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

Stage 3

State	Optimal Values	Optimal Decisions	Resulting State
4	25.30	3	7
5	18.60	3	8
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8
	$f_3(S_3)$	X_3^*	S_2^*

Stage 4

S_3	$f_3(S_3)$
4	25.30
5	18.60
6	12.50
7	6.70
8	0.00

Cost per plant = C_4 = 5.8

Req'd number of plants = R_4 = 4

$$f_4(S_4) = \underset{(R_4-S_4)^+\le X_4\le 3}{\textbf{minimum}} \; g_4(S_4,X_4)+f_3(S_4+X_4)$$

s \ x:	0	1	2	3
2	999999.99	999999.99	38.40	37.50
3	999999.99	32.60	31.70	31.40
4	25.30	25.90	25.60	25.60
5	18.60	19.80	19.80	18.90
6	12.50	14.00	13.10	999999.99
7	6.70	7.30	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

Stage 4

S \ X:	0	1	2	3
2	999999.99	999999.99	38.40	37.50
3	999999.99	32.60	31.70	31.40
4	25.30	25.90	25.60	25.60
5	18.60	19.80	19.80	18.90
6	12.50	14.00	13.10	999999.99
7	6.70	7.30	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

State	Optimal Values	Optimal Decisions	Resulting State
2	37.50	3	5
3	31.40	3	6
4	25.30	0	4
5	18.60	0	5
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8
	$f_4(S_4)$	X_4^*	S_3^*

S_4	$f_4(S_4)$
2	37.50
3	31.40
4	25.30
5	18.60
6	12.50
7	6.70
8	0.00

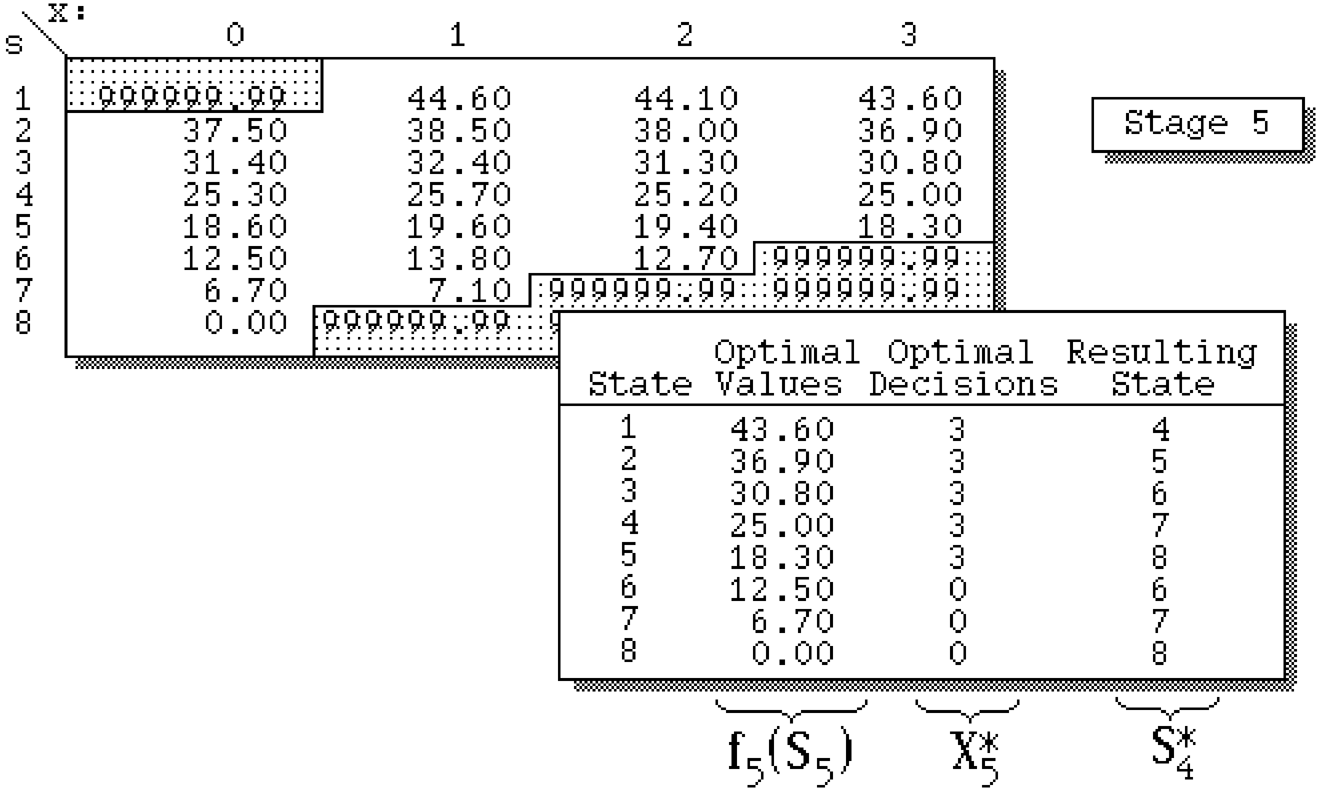
Stage 5

Cost per plant = C_5 = 5.6

Req'd number of plants = R_5 = 2

$$f_5(S_5) = \underset{(R_5 - S_5)^+ \le X_5 \le 3}{\text{minimum}} \; g_5(S_5, X_5) + f_4(S_5 + X_5)$$

s \ x:	0	1	2	3
1	999999.99	44.60	44.10	43.60
2	37.50	38.50	38.00	36.90
3	31.40	32.40	31.30	30.80
4	25.30	25.70	25.20	25.00
5	18.60	19.60	19.40	18.30
6	12.50	13.80	12.70	999999.99
7	6.70	7.10	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99



Stage 5

s \ x:	0	1	2	3
1	999999.99	44.60	44.10	43.60
2	37.50	38.50	38.00	36.90
3	31.40	32.40	31.30	30.80
4	25.30	25.70	25.20	25.00
5	18.60	19.60	19.40	18.30
6	12.50	13.80	12.70	999999.99
7	6.70	7.10	999999.99	999999.99
8	0.00	999999.99		

State	Optimal Values	Optimal Decisions	Resulting State
1	43.60	3	4
2	36.90	3	5
3	30.80	3	6
4	25.00	3	7
5	18.30	3	8
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

$f_5(S_5)$ X_5^* S_4^*

S_5	$f_5(S_5)$
1	43.60
2	36.90
3	30.80
4	25.00
5	18.30
6	12.50
7	6.70
8	0.00

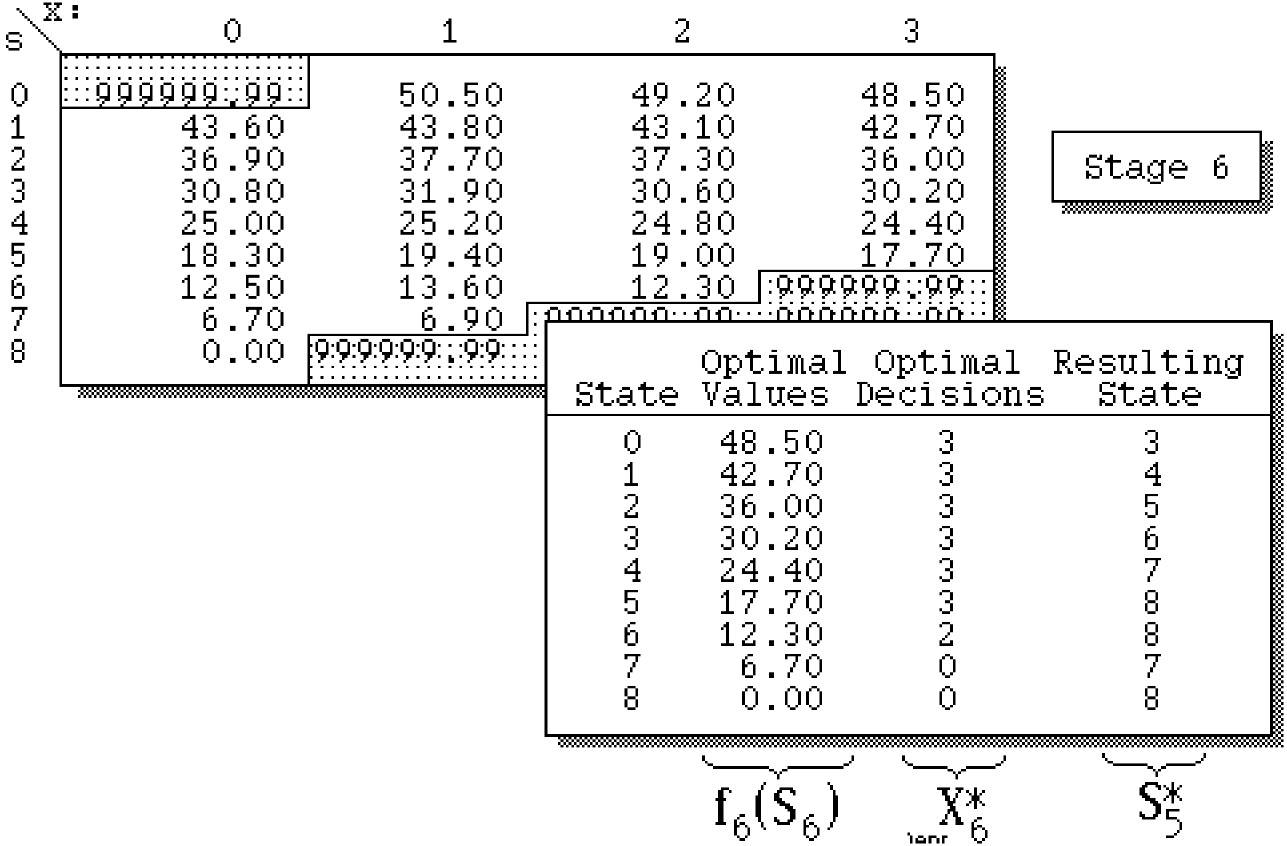
Stage 6

Cost per plant = C_6 = 5.4

Req'd number of plants = R_6 = 1

$$f_6(S_6) = \underset{(R_6-S_6)^+ \le X_6 \le 3}{\textbf{minimum}} \; g_6(S_6,X_6) + f_5(S_6+X_6)$$

s \ x:	0	1	2	3
0	999999.99	50.50	49.20	48.50
1	43.60	43.80	43.10	42.70
2	36.90	37.70	37.30	36.00
3	30.80	31.90	30.60	30.20
4	25.00	25.20	24.80	24.40
5	18.30	19.40	19.00	17.70
6	12.50	13.60	12.30	999999.99
7	6.70	6.90	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99



s \ x:	0	1	2	3
0	999999.99	50.50	49.20	48.50
1	43.60	43.80	43.10	42.70
2	36.90	37.70	37.30	36.00
3	30.80	31.90	30.60	30.20
4	25.00	25.20	24.80	24.40
5	18.30	19.40	19.00	17.70
6	12.50	13.60	12.30	999999.99
7	6.70	6.90	999999.99	999999.99
8	0.00	999999.99		

Stage 6

State	Optimal Values	Optimal Decisions	Resulting State
0	48.50	3	3
1	42.70	3	4
2	36.00	3	5
3	30.20	3	6
4	24.40	3	7
5	17.70	3	8
6	12.30	2	8
7	6.70	0	7
8	0.00	0	8

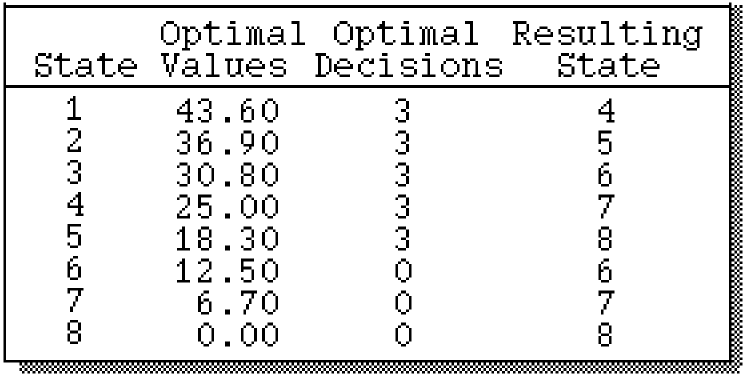
$f_6(S_6)$ X_6^* S_5^*

Optimal Returns & Decisions

Stage 6:

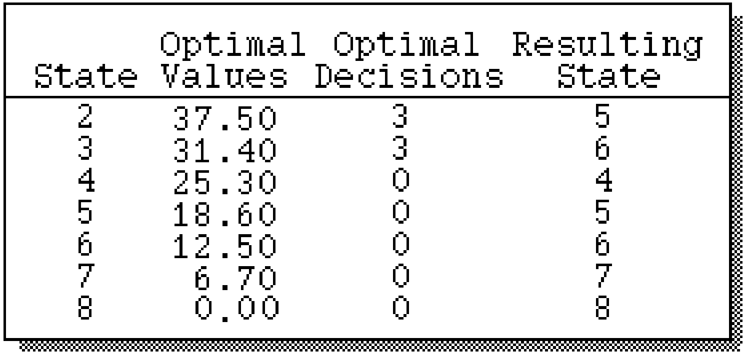
State	Optimal Values	Optimal Decisions	Resulting State
0	48.50	3	3
1	42.70	3	4
2	36.00	3	5
3	30.20	3	6
4	24.40	3	7
5	17.70	3	8
6	12.30	2	8
7	6.70	0	7
8	0.00	0	8

Stage 5:



State	Optimal Values	Optimal Decisions	Resulting State
1	43.60	3	4
2	36.90	3	5
3	30.80	3	6
4	25.00	3	7
5	18.30	3	8
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

Stage 4:



State	Optimal Values	Optimal Decisions	Resulting State
2	37.50	3	5
3	31.40	3	6
4	25.30	0	4
5	18.60	0	5
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

Stage 3:

State	Optimal Values	Optimal Decisions	Resulting State
4	25.30	3	7
5	18.60	3	8
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

Stage 2:

State	Optimal Values	Optimal Decisions	Resulting State
6	12.50	2	8
7	6.70	0	7
8	0.00	0	8

Stage 1:

State	Optimal Values	Optimal Decisions	Resulting State
7	6.70	1	8
8	0.00	0	8

Optimal Solution

The initial state (entering stage 6) is 0

Power Plant Capacity Planning

*** Optimal value is 48.5 ***

STAGE	STATE	DECISION
6	0	3
5	3	3
4	6	0
3	6	0
2	6	2
1	8	0
0	8	

Discounting Future Costs

Let r = rate of interest

Then

$1 invested now has the same value as

$\$1(1+r)^n$ after n periods,

and

$1 paid n periods hence is equivalent to

$$\$1\frac{1}{(1+r)^n} = \$1\,\beta^n \quad \text{where } \beta = \frac{1}{1+r}$$

discount factor

paid now.

Minimizing total discounted future costs, i.e., present value of future costs, in a DP model:

In the power plant capacity planning model:

$$f_n(S_n) = \begin{cases} \underset{(R_n - S_n)^+ \le X_n \le 3}{\text{minimum}} \left\{ g_n(S_n, X_n) + \beta f_{n-1}(S_n + X_n) \right\} & \text{if } n \ge 1 \\ 0 & \text{if } n=0 \;\&\; S_0 \ge R_1 \\ +\infty & \text{if } n=0 \;\&\; S_0 < R_1 \end{cases}$$

discount factor

= minimum present value of costs of last n stages

("present" ⟷ stage n)

Power Plant Capacity Planning

State Vector

i	1	2	3	4	5	6	7	8	9
s[i]	0	1	2	3	4	5	6	7	8

Decision Vector

i	1	2	3	4
x[i]	0	1	2	3

Suppose that $\mathbf{r} = 20\%$, so that $\beta = \frac{1}{1.2} = 0.83333$

Power Plant Capacity Planning

```
z←F N;t;S;G
⍝
⍝             Optimal Value Function
⍝         Power Plant Capacity Planning problem
⍝         (minimizing present value of future costs, with
⍝          discount factor beta)
⍝
→End IF N=0

⍝       Cost of adding plants in current year
G←((⍴s)⍴0)∘.+(x>0)×1.5+COST[N]×x

⍝       Cumulative # of plants added at beginning of next yr
S←s∘.+x
⍝       Recursive definition of optimal value
z← MIN G + beta×(F N-1)[TRANSITION S+BIG×(S<RQMT[N])]
→0
End:z←((⍴s)⍴0),BIG
```

Note: recursion is backward

APL Code

Power Plant Capacity Planning

```
Recursion type: backward

                                                  ——Stage 1——
 s \ x:   0         1         2         3
 -   -- --  -         -         -         -

 5 | 9999.99   9999.99   9999.99     17.10
 6 | 9999.99   9999.99     11.90   9999.99
 7 | 9999.99      6.70   9999.99   9999.99
 8 |    0.00   9999.99   9999.99   9999.99
```

```
         Optimal    Optimal     Resulting
 State   Values    Decisions     State

   5      17.10        3           8
   6      11.90        2           8
   7       6.70        1           8
   8       0.00        0           8
```

$f_1(S_1)$

```
s \ x:  0        1        2        3

4 | 9999.99  9999.99  9999.99    23.58
5 | 9999.99  9999.99    18.08    18.00
6 | 9999.99    12.58    12.50  9999.99
7 |    5.58     7.00  9999.99  9999.99
8 |    0.00  9999.99  9999.99  9999.99
```

S_1	$f_1(S_1)$
5	17.10
6	11.90
7	6.70
8	0.00

```
---Stage 2---

         Optimal   Optimal     Resulting
State    Values    Decisions    State
  4      23.58        3           7
  5      18.00        3           8
  6      12.50        2           8
  7       5.58        0           7
  8       0.00        0           8
```

$\leftarrow f_2(S_2)$

```
s \ x:  0         1         2         3

3 | 9999.99  9999.99  9999.99    29.02
4 | 9999.99  9999.99    23.32    23.25
5 | 9999.99    17.62    17.55    18.60
6 |   10.42    11.85    12.90  9999.99
7 |    4.65     7.20  9999.99  9999.99
8 |    0.00  9999.99  9999.99  9999.99
```

S_2	$f_2(S_2)$
4	23.58
5	18.00
6	12.50
7	5.58
8	0.00

---Stage 3---

```
         Optimal  Optimal     Resulting
State    Values   Decisions    State
  3      29.02       3           6
  4      23.25       3           7
  5      17.55       2           7
  6      10.42       0           6
  7       4.65       0           7
  8       0.00       0           8
```

$f_3(S_3)$

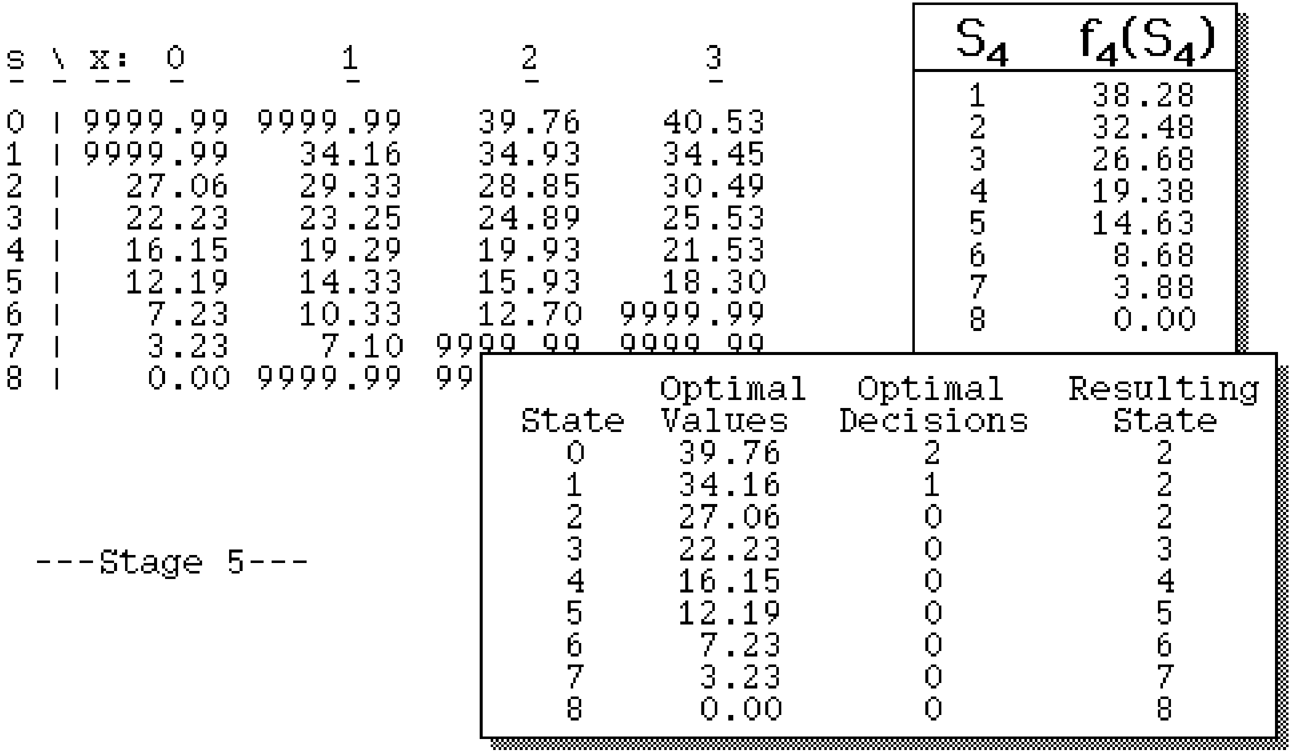
```
s \ x:  0        1        2        3
-  -  --  -        -        -        -

1 |  9999.99  9999.99  9999.99    38.28
2 |  9999.99  9999.99    32.48    33.53
3 |  9999.99    26.68    27.73    27.58
4 |    19.38    21.93    21.78    22.78
5 |    14.63    15.98    16.98    18.90
6 |     8.68    11.18    13.10  9999.99
7 |     3.88     7.30  9999.99  9999.99
8 |     0.00  9999.99  9999.99  9999.99
```

S_3	$f_3(S_3)$
3	29.02
4	23.25
5	17.55
6	10.42
7	4.65
8	0.00

---Stage 4---

State	Optimal Values	Optimal Decisions	Resulting State
1	38.28	3	4
2	32.48	2	4
3	26.68	1	4
4	19.38	0	4
5	14.63	0	5
6	8.68	0	6
7	3.88	0	7
8	0.00	0	8

$f_4(S_4)$



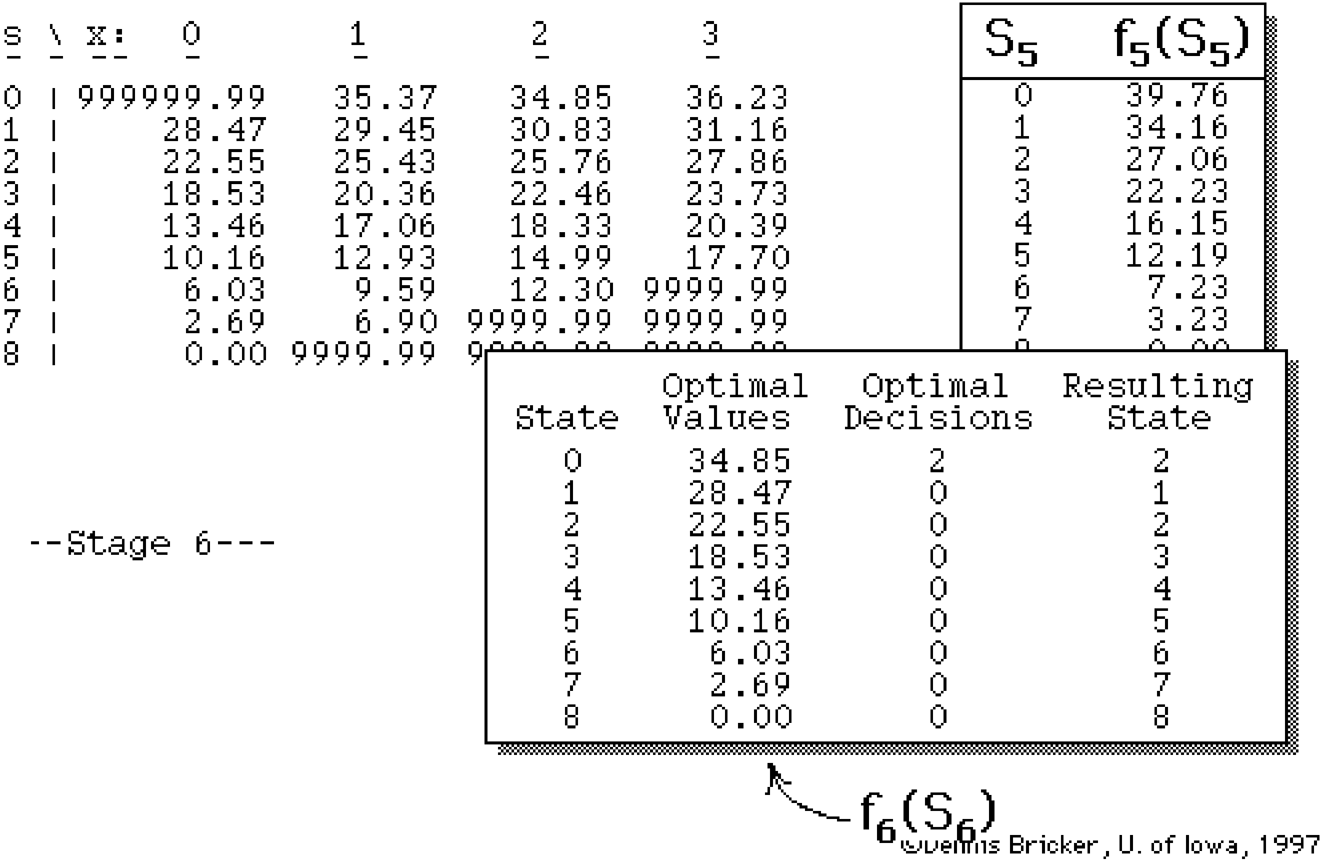
```
s \ x:  0        1        2        3
-   -   --  -        -        -        -

0 | 9999.99  9999.99    39.76    40.53
1 | 9999.99    34.16    34.93    34.45
2 |   27.06    29.33    28.85    30.49
3 |   22.23    23.25    24.89    25.53
4 |   16.15    19.29    19.93    21.53
5 |   12.19    14.33    15.93    18.30
6 |    7.23    10.33    12.70  9999.99
7 |    3.23     7.10  9999.99  9999.99
8 |    0.00  9999.99  99
```

S_4	$f_4(S_4)$
1	38.28
2	32.48
3	26.68
4	19.38
5	14.63
6	8.68
7	3.88
8	0.00

```
---Stage 5---
```

State	Optimal Values	Optimal Decisions	Resulting State
0	39.76	2	2
1	34.16	1	2
2	27.06	0	2
3	22.23	0	3
4	16.15	0	4
5	12.19	0	5
6	7.23	0	6
7	3.23	0	7
8	0.00	0	8

$f_5(S_5)$



```
s \ x:   0          1          2          3
-  - --  -          -          -          -

0 | 999999.99    35.37      34.85      36.23
1 |     28.47    29.45      30.83      31.16
2 |     22.55    25.43      25.76      27.86
3 |     18.53    20.36      22.46      23.73
4 |     13.46    17.06      18.33      20.39
5 |     10.16    12.93      14.99      17.70
6 |      6.03     9.59      12.30    9999.99
7 |      2.69     6.90    9999.99    9999.99
8 |      0.00  9999.99    9
```

S_5	$f_5(S_5)$
0	39.76
1	34.16
2	27.06
3	22.23
4	16.15
5	12.19
6	7.23
7	3.23

--Stage 6---

State	Optimal Values	Optimal Decisions	Resulting State
0	34.85	2	2
1	28.47	0	1
2	22.55	0	2
3	18.53	0	3
4	13.46	0	4
5	10.16	0	5
6	6.03	0	6
7	2.69	0	7
8	0.00	0	8

$f_6(S_6)$

Power Plant Capacity Planning

*** Optimal value is 34.85369084 ***

STAGE	STATE	DECISION
6	0	2
5	2	0
4	2	2
3	4	3
2	7	0
1	7	1
0	8	