Covering Problems in Facility Location

- **Minimum cost covering**
  covering is mandatory,
  number or cost of facilities is to be minimized

- **Maximal cover**
  number or cost of facilities is fixed
  amount of coverage is to be maximized

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GIVEN DATA:

m customers to be covered, n candidate locations

$$a_{ij} = \begin{cases} 1 & \text{if facility at } j \text{ covers customer } i \\ 0 & \text{otherwise} \end{cases}$$

$$C_j = \text{cost of set } j$$

DECISION VARIABLES:

$$X_j = \begin{cases} 1 & \text{if set } j \text{ is included in the cover} \\ 0 & \text{otherwise} \end{cases}$$

The Minimum Cost Covering Problem makes coverage mandatory, and uses the standard **S**et **C**overing **P**roblem **(SCP)** model:

- Minimize the cost (or number) of facilities

- Require that every customer be covered *at least once*

This model is often applied to the location of *public* facilities, where a certain *level of service* must be guaranteed to all "customers", as in

- hospitals
- fire stations
- ambulances
- schools

Given: C_j = cost of facility j (1 if minimizing the *number* of facilities)

$$a_{ij} = \begin{cases} 1 & \text{if customer } i \text{ is covered by facility } j \\ 0 & \text{otherwise} \end{cases}$$

Often a_{ij} is defined in terms of a distance parameter S:

$$a_{ij} = \begin{cases} 1 & \text{if distance from facility } j \text{ to customer } i \text{ is } \le S \\ 0 & \text{otherwise} \end{cases}$$

In this case, **distance** must be clearly defined:

- distance in network
- travel time in network
- Euclidean distance
- Rectilinear *(Manhattan)* distance

  etc.

Define decision variables:

$$X_j = \begin{cases} 1 & \text{if facility } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

SCP: *Minimize* $\sum_{j=1}^{n} \mathrm{C_j} X_j$

subject to:

for all $i = 1,2,\ldots m,\ \sum_{j=1}^{n} a_{ij} X_j \geq 1$

for all $j = 1,2,\ldots n,\quad X_j \in \{0,1\}$

The **M**aximal **C**overage **L**ocation **P**roblem **(MCLP)**

- the cost or number of facilities is restricted
- coverage of customers (or demand) is not mandatory, but is maximized

This type of model is most often applied to profit-seeking or ***private*** enterprises.

Decision variables:

$$X_j = \begin{cases} 1 & \text{if set } j \text{ is included in the cover} \\ 0 & \text{otherwise} \end{cases}$$

$$Y_i = \begin{cases} 1 & \text{if coverage of customer } i \text{ is required} \\ 0 & \text{otherwise} \end{cases}$$

Given Data:

m customers to be covered, n candidate locations

$$a_{ij} = \begin{cases} 1 & \text{if facility at } j \text{ covers customer } i \\ 0 & \text{otherwise} \end{cases}$$

$C_j =$ cost of set j

$B =$ capital budget

$D_j =$ demand generated by (or population of) customer i

The **budget constraint** is $\sum_{j=1}^{n} C_j X_j \le B$

or, if we are restricting the *number* of facilities to p,

$$\sum_{j=1}^{n} X_j \le p$$

i.e., $C_j = 1$ and $B = p$.

The **coverage constraint** is

for each customer $i = 1, 2, \ldots m$, $\sum_{j=1}^{n} a_{ij} X_j \ge Y_i$

The **objective** is maximize $\sum_{i=1}^{m} D_i Y_i$

or, if we are maximizing the number of customers covered,

maximize $\sum_{i=1}^{m} Y_i$, i.e., $D_i = 1$.

MCLP: Maximize $\sum_{i=1}^{m} D_i Y_i$

subject to

$$\sum_{j=1}^{n} C_j X_j \le B$$

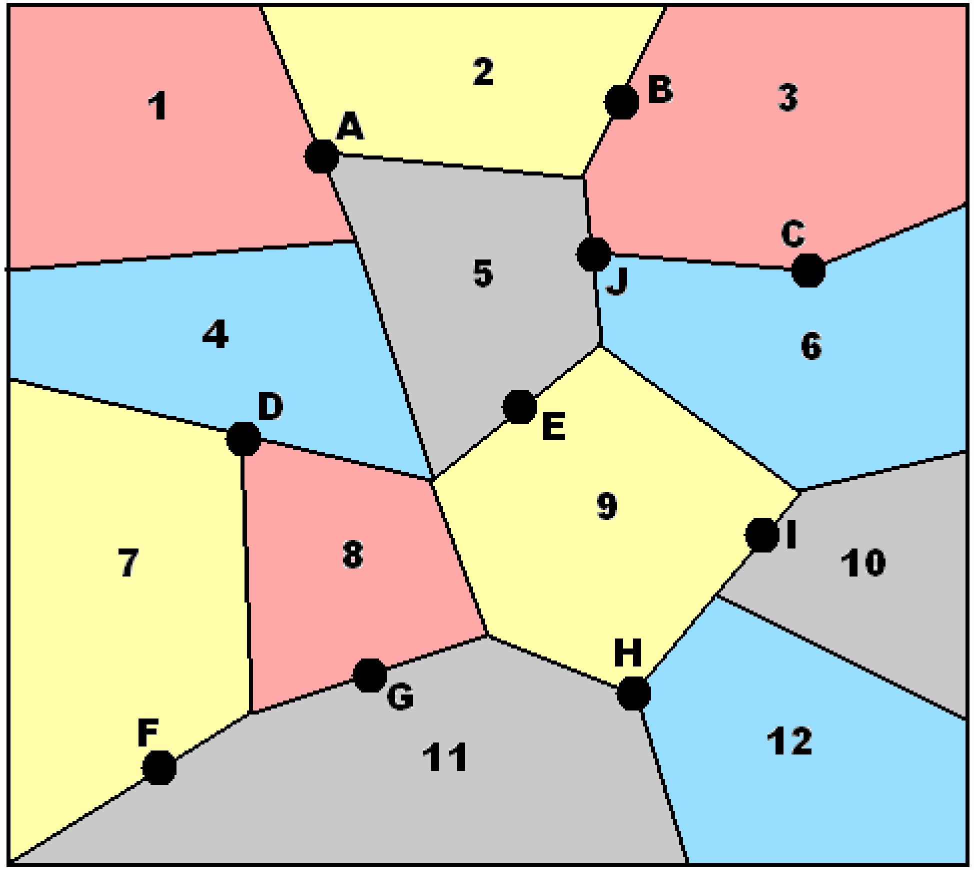
$$\sum_{j=1}^{n} a_{ij} X_j \ge Y_i \quad \text{for all } i = 1,2,\ldots m,$$

$$X_j \in \{0,1\} \qquad \text{for all } j = 1,2,\ldots n,$$

$$Y_i \in \{0,1\} \qquad \text{for all } i = 1,2,\ldots m$$

Other variations:

- Finite facility ***capacity***, restricting the amount of demand that a facility may serve.
- Number of customers to be ***double-served*** is included in objective, especially in order to *break ties* among multiple optimal solutions.
- ***Differentiation***, i.e., the facilities provide an identical service or product.



A mobile phone operator wants to provide service to a currently uncovered geographical region. Seven locations are being considered for installation for towers for this purpose.

Because of the distances and obstacles such as mountains and tall buildings, each tower can serve only a small number of the twelve communities in the region. Furthermore, the costs of building towers depends upon the site.

The following table gives the following information: an "X" indicating that a tower covers a community; the population (in thousands) of each community; the cost (in US$ millions) of purchasing the land and building the tower.

site→ community↓	A	B	C	D	E	F	G	H	I	J	Pop. (K)
1	X			X							2
2	X	X								X	4
3		X	X							X	3
4				X			X				3
5	X			X	X					X	5
6			X		X				X	X	6
7				X		X	X				2
8				X	X		X				7
9					X		X	X	X		6
10								X	X		5
11						X	X	X			4
12							X	X	X		3
Cost (US$M)	0.8	0.6	0.9	1.4	1.4	1.2	1.6	0.9	0.8	0.8	

SCP: The mobile telephone company wishes to cover all the communities at the lowest cost of constructing the towers.

Where should be towers be placed?

MCLP: Suppose that the telephone company has a budget of US$3 million and wants to provide service to as many customers as possible.

Where should the towers be placed?

```
MODEL:   ! Set covering problem - LINGO model #1;

SETS:
   COMMUNITY/1..12/:POP;             ! Communities to be covered;
   TOWER/1..10/:COST,X;              ! Towers which can be built;
   COVER(COMMUNITY,TOWER):A;
ENDSETS

DATA:
   POP= 2 4 3 3 5 6 2 7 6 4 4 3;    ! Population of communities;
   COST= 0.8 .6 0.9 1.4 1.4 1.2 1.6 0.9 0.8 0.8;  ! Cost of towers;
   A =  1 0 0 1 0 0 0 0 0 0   ! Coverage matrix ;
        1 1 0 0 0 0 0 0 0 1   !(row per community, column per tower);
        0 1 1 0 0 0 0 0 0 1
        0 0 0 1 0 0 1 0 0 0
        1 0 0 1 1 0 0 0 0 1
        0 0 1 0 1 0 0 0 1 1
        0 0 0 1 0 1 1 0 0 0
        0 0 0 1 1 0 1 0 0 0
        0 0 0 0 1 0 1 1 1 0
        0 0 0 0 0 0 0 1 1 0
        0 0 0 0 0 1 1 1 0 0
        0 0 0 0 0 0 1 1 1 0;
ENDDATA
```

```
MIN = @SUM(TOWER(J):  COST(J)*X(J) );

@FOR(COMMUNITY(I):
    @SUM(TOWER(J):  A(I,J)*X(J) ) >= 1;
     );

@FOR(TOWER(J):   @BIN(X(J));              ! Specify that X is binary;
     );

END
```

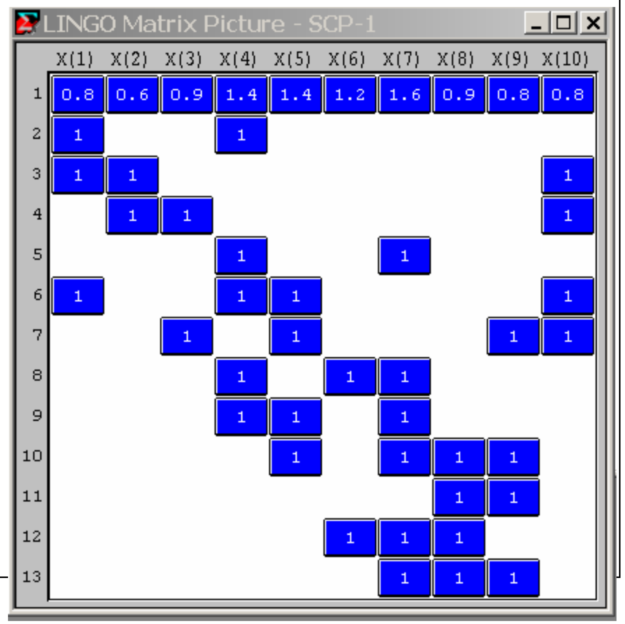
```
Global optimal solution found:
Objective value:          3.100000

Variable        Value        Reduced Cost
X( 4)        1.000000        1.400000
X( 8)        1.000000        0.9000000
X(10)        1.000000        0.8000000
```

An alternative model which allows a specification of the coverage matrix which is less prone to error:

```
MODEL:    ! Set covering problem - LINGO model #2;

SETS:
   COMMUNITY/1..12/:POP;      ! Communities to be covered;
   TOWER/1..10/:COST,X;       ! Towers which can be built;
   COVER(COMMUNITY,TOWER)/    ! List of elements in Cover matrix;
   1,1    1,4                 ! Less error-prone than previous method;
   2,1    2,2   2,10
   3,1    3,2   3,10
   4,4    4,7
   5,1    5,4   5,5    5,10
   6,3    6,5   6,9    6,10
   7,4    7,6   7,7
   8,4    8,5   8,7
   9,5    9,7   9,8    9,9
  10,8   10,9
  11,6   11,7  11,8
  12,7   12,8  12,9/;
ENDSETS
```



```
DATA:
   POP= 2 4 3 3 5 6 2 7 6 4 4 3;    ! Population of communities;
   COST= 0.8 .6 0.9 1.4 1.4 1.2 1.6 0.9 0.8 0.8;  ! Cost of towers;
ENDDATA

MIN = @SUM(TOWER(J):  COST(J)*X(J) );

@FOR(COMMUNITY(I):
    @SUM(COVER(I,J): X(J) ) >= 1;
     );

@FOR(TOWER(J):   @BIN(X(J));         !  Specify that X is binary;
     );

END
```

II. **Suppose that the telephone company has a budget of US$3 million and wants to provide service to as many customers as possible. Where should the towers be placed?**

Define **decision variables**:

For each $j = 1,2,...10$:

$$X_j = \begin{cases} 1 & \text{if tower } j \text{ is built} \\ 0 & \text{otherwise} \end{cases}$$

For each $i = 1,2,...12$:

$$Z_i = \begin{cases} 1 & \text{if community } i \text{ is covered} \\ 0 & \text{otherwise} \end{cases}$$

Maximum Cover Problem

Objective:

$$\text{Maximize} \quad \sum_{i=1}^{12} P_i Z_i$$

Constraints:

For each community $i = 1,2,\ldots 12$:

$$Z_i \leq \sum_{j=1}^{10} a_{ij} X_j$$

This forces $Z_i = 0$ if $\sum_{j=1}^{10} a_{ij} X_j = zero$.

Because of the objective, Z_i will be as large as possible,

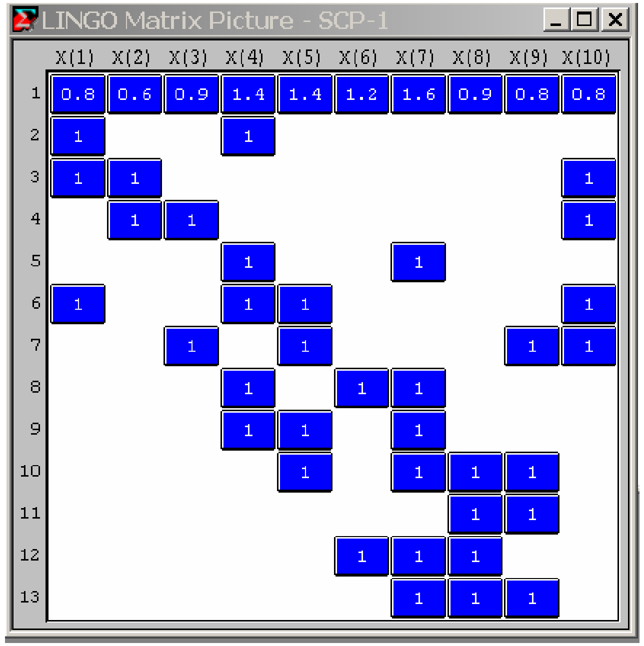
i.e, 1, if $\sum_{j=1}^{10} a_{ij} X_j \geq 1$.

For all i & j: $Z_i \in \{0,1\}$ & $X_j \in \{0,1\}$

```
MODEL:   ! Maximum covering problem;

SETS:
   COMMUNITY/1..12/:POP,Z;          ! Communities;
   TOWER/1..10/:COST,X;             ! Towers which can be built;
   COVER(COMMUNITY,TOWER)/          ! List of elements in Cover matrix;
   1,1    1,4
   2,1    2,2   2,10
   3,1    3,2   3,10
   4,4    4,7
   5,1    5,4   5,5    5,10
   6,3    6,5   6,9    6,10
   7,4    7,6   7,7
   8,4    8,5   8,7
   9,5    9,7   9,8    9,9
  10,8   10,9
  11,6   11,7  11,8
  12,7   12,8  12,9/;
ENDSETS

DATA:
   POP= 2 4 3 3 5 6 2 7 6 4 4 3;     ! Population of communities;
   COST= 0.8 .6 0.9 1.4 1.4 1.2 1.6 0.9 0.8 0.8;  ! Cost of towers;
ENDDATA
```



```
BUDGET = 3;

MAX = @SUM(COMMUNITY(I):   POP(I)*Z(I) );

@SUM(TOWER(J):  COST(J)*X(J) ) <= BUDGET ;

@FOR(COMMUNITY(I):
    @SUM(COVER(I,J): X(J) ) >= Z(I);
);
                ! Specify that X &  Z are binary variables;
@FOR(TOWER(J):   @BIN(X(J));
     );
@FOR(COMMUNITY(I): @BIN(Z(I));
     );

END
```

The generated model:

```
MAX      2 Z( 1) + 4 Z( 2) + 3 Z( 3) + 3 Z( 4) + 5 Z( 5) + 6 Z( 6)
      + 2 Z( 7) + 7 Z( 8) + 6 Z( 9) + 4 Z( 10) + 4 Z( 11) + 3 Z( 12)
 SUBJECT TO
 2]  .8 X( 1) + .6 X( 2) + .9 X( 3) + 1.4 X( 4) + 1.4 X( 5) + 1.2 X( 6)
      + 1.6 X( 7) + .9 X( 8) + .8 X( 9) + .8 X( 10) <=   3
 3]  X( 1) + X( 4) - Z( 1) >=   0
 4]  X( 1) + X( 2) + X( 10) - Z( 2) >=   0
 5]  X( 1) + X( 2) + X( 10) - Z( 3) >=   0
 6]  X( 4) + X( 7) - Z( 4) >=   0
 7]  X( 1) + X( 4) + X( 5) + X( 10) - Z( 5) >=   0
 8]  X( 3) + X( 5) + X( 9) + X( 10) - Z( 6) >=   0
 9]  X( 4) + X( 6) + X( 7) - Z( 7) >=   0
 10]  X( 4) + X( 5) + X( 7) - Z( 8) >=   0
 11]  X( 5) + X( 7) + X( 8) + X( 9) - Z( 9) >=   0
 12]  X( 8) + X( 9) - Z( 10) >=   0
 13]  X( 6) + X( 7) + X( 8) - Z( 11) >=   0
 14]  X( 7) + X( 8) + X( 9) - Z( 12) >=   0
 END
 INTE    22
```

The optimal solution:

```
Global optimal solution found at step:  79
 Objective value:                  45.00000
 Branch count:                            1
                      Variable           Value        Reduced Cost
                         X( 2)        1.000000           0.0000000
                         X( 4)        1.000000           0.0000000
                         X( 9)        1.000000           0.0000000

                      Variable           Value        Reduced Cost
                         Z( 1)        1.000000           -2.000000
                         Z( 2)        1.000000           -4.000000
                         Z( 3)        1.000000           -3.000000
                         Z( 4)        1.000000           -3.000000
                         Z( 5)        1.000000           -5.000000
                         Z( 6)        1.000000           -6.000000
                         Z( 7)        1.000000           -2.000000
                         Z( 8)        1.000000           -7.000000
                         Z( 9)        1.000000           -6.000000
                        Z( 10)        1.000000           -4.000000
                        Z( 12)        1.000000           -3.000000
```

Actual cost is only 2.8 $M. All communities can be covered except #11!

Often there are ***multiple*** optimal solutions...

but optimizers will report ***only one*** (selected arbitrarily).

In this case, we would sometimes like to add another criterion to break ties among these optimal solutions, e.g.,

- maximize the number of demand nodes covered twice.

  This will add **reliability** to the solution *(facilities might become temporarily unavailable!)*
- maximize the use of existing facilities

  if some of the facilities already exist.

Secondary Objective:

Maximize the number of demand nodes covered twice

Define another set of **decision variables**:

For all $i = 1, 2, \ldots m$: $Y_i = \begin{cases} 1 & \text{if demand } i \text{ is covered } \textit{at least} \text{ twice} \\ 0 & \text{otherwise} \end{cases}$

Replace each constraint

$\sum_{j=1}^{n} a_{ij} X_j \geq 1$ with the constraint $\sum_{j=1}^{n} a_{ij} X_j \geq 1 + Y_i$

We want to modify the objective $\min \sum_{j=1}^{n} \mathrm{X_j}$ so that:

of two solutions having the same value of $\min \sum_{j=1}^{n} \mathrm{X_j}$,

the one with larger $\sum_{i=1}^{m} Y_i$ *will have a lower cost!*

Therefore,

we subtract some multiple of $\sum_{i=1}^{m} Y_i$ which cannot be larger than the cost of adding another facility to the cover.

Since $\sum_{i=1}^{m} Y_i < m+1 \quad \Leftrightarrow \left(\sum_{i=1}^{m} Y_i \middle/ m+1 \right) < 1,$

we subtract $\left(\sum_{i=1}^{m} Y_i \middle/ m+1 \right)$ from $\sum_{j=1}^{n} X_j$ as a *"tie-breaker"*.

That is, $\min \sum_{j=1}^{n} \mathrm{X}_j$ is replaced by $\min \sum_{j=1}^{n} \mathrm{X}_j - \sum_{i=1}^{m} Y_i \big/ m+1$

This added term is always less than the cost of adding another set to the cover!

Secondary objective:

Maximize the use of any facilities already existing.

Define the subset J_E of facilities to be those already existing.

*We want there to be some **disincentive** ε to selecting a non-existing facility instead of an existing one.*

Replace the objective $\min \sum_{j=1}^{n} \mathrm{X_j}$ by $\min \sum_{j \in J_E} \mathrm{X_j} + (1+\varepsilon) \sum_{j \notin J_E} \mathrm{X_j}$

This disincentive ε should never be more than $1/|J_E|$, where $|J_E|$ is the number of existing facilities in J_E.