Revised Simplex Method for LP

In the "ordinary" simplex method, the *full* tableau is updated at every iteration, but much of the information in the tableau is not used in each iteration-- e.g., only the substitution rates in the column entering the basis are actually used!

The "revised" simplex method computes *only* those numbers in the tableau that are required at the current iteration.

Advantages:

- savings in storage requirements in memory
- savings in computation
- control of round-off errors

Basis Set

The set of indices, i.e., column numbers, of the basic variables forms the **basis**, denoted B

For example,
if X_2, X_7, and X_3 are basic variables, then

$$B = \{2, 7, 3\}$$

Basic Variables

The basic variables corresponding to the basis B = {3,6,1} form a subvector of X, denoted by

$$X_B = \{ X_3, X_6, X_1 \}$$

The subvector of the cost vector C corresponding to this basis is denoted

$$C_B = \{ C_3, C_6, C_1 \}$$

Basis Matrix

The columns of the constraint coefficient matrix **A** indexed by the basis B is the **basis matrix** denoted A^B

For example, if B={3,6,1} and the original tableau is

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

then

$$A^B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Basis Inverse

The inverse of the basis matrix will be called the **basis inverse**

For example, if B={3,6,1} and $A^B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

then the basis inverse is

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

In the simplex method, we pivot in the tableau in order to change the basis:

original tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	(3)	2	-1	0	0	5
0	5	0	2	9	0	(1)	0	8
0	(3)	-1	0	5	0	0	1	3

tableau with basic variables $-Z, X_3, X_6, X_1$

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	0	3.22	0	2.56	0.333	0	‾0.556	‾3.33
0	0	0.444	1	0.111	‾0.333	0	‾0.111	1.33
0	0	0.778	0	0.444	0.667	1	‾1.44	0.333
0	1	‾0.333	0	1.67	0	0	0.333	1

If we have the basis inverse matrix and the original tableau, we can compute the current tableau:

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 3 & 2 & -1 & 0 & 0 \\ 5 & 0 & 2 & 9 & 0 & 1 & 0 \\ 3 & -1 & 0 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$(A^B)^{-1}A = \begin{bmatrix} 0 & 0.444 & 1 & 0.111 & -0.333 & 0 & -0.111 \\ 0 & 0.778 & 0 & 0.444 & 0.667 & 1 & -1.44 \\ 1 & -0.333 & 0 & 1.67 & 0 & 0 & 0.333 \end{bmatrix}$$

which is that section of the current tableau which contained A originally!

If the variable X_k has been chosen to enter the basis, then the "minimum ratio test" must be performed to choose the pivot row.

The "**substitution rates**" to be used in the denominator of these ratios may therefore be obtained by $(A^B)^{-1}A^k$.

e.g.,

$$\alpha = (A^B)^{-1}A^4 = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 4/9 \\ 5/3 \end{bmatrix}$$

We can use the basis inverse matrix to compute the values of the basic variables:

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$$(A^B)^{-1} b = \begin{bmatrix} 1.33 \\ 0.333 \\ 1 \end{bmatrix}$$

which is the RHS of the current tableau, i.e., the values of the basic variables:

$$\begin{cases} X_3 = 4/3 \\ X_6 = 1/3 \\ X_1 = 1 \end{cases}$$

Note the order, which corresponds to $B = \{3,6,1\}$

The **reduced cost** of variable X_k is

$$\overline{C}_k = C_k - \sum_{i=1}^{m} C_{B_i}\alpha_i$$

where α_i is the substitution rate of X_k for the basic variable in row i, i.e., of X_{B_i}

That is, if X_k *is increased by 1 unit,* C_k *is added to the cost, while the basic variable in row i is reduced by the amount* α_i*, thereby saving (if* $\alpha_i > 0$*) a cost* $C_{B_i}\alpha_i$

Since $\alpha = (A^B)^{-1}A^k$, $\overline{C}_k = C_k - C_B(A^B)^{-1}A^k$

If we define $\pi = C_B(A^B)^{-1}$, then

$$\overline{C}_k = C_k - \pi A^k = C_k - \sum_{i=1}^{m} \pi_i A_i^k$$

original tableau:

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$B = \{3,6,1\}$

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\pi = C_B(A^B)^{-1} = [1, 0, 2] \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} = [1/3, 0, 5/9]$$

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$B = \{3,6,1\}$

$\pi = [1/3, 0, 5/9]$

Suppose we want to "price" the nonbasic variable X_4:

$$\overline{C}_4 = C_4 - \pi A^4 = 6 - \left(\frac{1}{3}\times 2 + 0\times 9 + \frac{5}{9}\times 5\right) = 23/9 = 2.555555$$

In the **Revised Simplex Method**, no pivoting is done in the LP tableau.

When an entry of the tableau is needed, it is computed, using the inverse of the basis matrix.

- relative profit (or reduced cost) of X_j is $C_j - \pi A^j$ where $\pi = C_B(A^B)^{-1}$
- substitution rates for the entering variable X_j (used in the minimum ratio test) are $\alpha = (A^B)^{-1}A^j$

Revised Simplex Method

Step 0: determine an initial feasible basis B

Step 1: compute the simplex multiplier vector $\pi = C_B(A^B)^{-1}$

Step 2: price the nonbasic columns, i.e., compute the reduced cost of each nonbasic variable

Step 3: select the entering variable X_k

Step 4: compute the substitution rates of X_k $\alpha = (A^B)^{-1}A^k$

Revised Simplex Method

Step 5: determine the variable to leave the basis

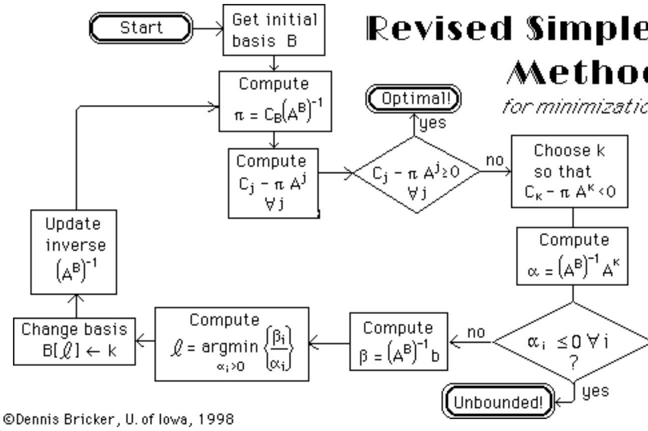
$$\min_{a_i>0}\left\{\frac{\beta_i}{\alpha_i}\right\} = \frac{\beta_\ell}{\alpha_\ell} \quad \text{where } \beta = X_B = \left(A^B\right)^{-1} b$$

Step 6: replace the ℓ^{th} index of B with k, and update the inverse matrix $\left(A^B\right)^{-1}$

Return to step 1.

Revised Simplex Method

for minimization



-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

B = {3,6,1}

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\pi = C_B\left(A^B\right)^{-1} = \left[1/3,\ 0,\ 5/9\right]$$

Example:

Pricing X_7: $\overline{C}_7 = C_7 - \pi A^7$

$$= 0 - \left(\frac{1}{3}\times 0 + 0\times 0 + \frac{5}{9}\times 1\right) = -5/9$$

So X_7 will enter the basis

original tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

B = {3,6,1}

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

current tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	0		0			0	-5/9	
0	0		1			0		
0	0		0			1		
0	1		0			0		

Since the reduced cost of the variable X_7 is negative, we will enter it into the basis.

To choose the pivot row will require the "substitution rates" in the column.

original tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

B = {3,6,1}

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Calculation of "substitution rates" $\alpha = \left(A^B\right)^{-1} A^j$

$$\alpha = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/9 \\ -13/9 \\ 1/3 \end{bmatrix}$$

current tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	0		0			0	-5/9	
0	0		1			0	-1/9	
0	0		0			1	-13/9	
0	1		0			0	1/3	

Next, to perform the "minimum ratio test" for selecting the pivot row, we need the current right-hand-side: $\beta = X_B = \left(A^B\right)^{-1} b$

original tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

B = {3,6,1}

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\beta = X_B = \left(A^B\right)^{-1} b = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 1 \end{bmatrix}$$

current tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	0		0			0	-5/9	
0	0		1			0	-1/9	2/3
0	0		0			1	-13/9	1/3
0	1		0			0	1/3	1

Now we can select the pivot row, which is the last row (since there is only one positive substitution rate). That is, X_7 enters the basis, replacing X_1, so that the new basis is B = {3,6,7} *original tableau*

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

B = {3,6,7}

$$A^B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rather than to compute the basis inverse matrix "from scratch", it is possible to update the old basis inverse matrix:

$$\underbrace{\begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}}_{\text{old inverse}} \Bigg\| \underbrace{\begin{bmatrix} -1/9 \\ -13/9 \\ \boxed{1/3} \end{bmatrix}}_{\text{pivot column}} \longrightarrow \underbrace{\begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{new inverse}} \Bigg\| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We are then ready to begin another iteration of the revised simplex method.

$$\pi = \left[-1/3\ ,\ 0,\ 0\right]$$

$$\bar{C}_2 = 3 - (\pi_1 - \pi_3) = 8/3$$

$$\bar{C}_4 = 6 - (2\pi_1 + 9\pi_2 + 5\pi_3) = 16/3$$

$$\bar{C}_5 = 0 - (-\pi_1) = 1/3$$

Since the reduced costs of the nonbasic variables are all positive, the current basic solution is optimal!

Advantages of Revised Simplex Method

- Because we pivot only in the basis inverse matrix, each pivot requires updating only mxm numbers, not (m+1)(n+1) (where m=#rows, n=#columns)

For example, if m=100 and n=1000 (not unusually big in practice), the RSM updates only 10,000 numbers in the basis inverse, while the ordinary simplex method updates 100,000 numbers in the tableau.

Advantages of Revised Simplex Method

- Most real-world LP problems have an original tableau which is very sparse, e.g., only 1% to 5% of the numbers are nonzero. This allows efficiency of storage in memory.
  After several pivots in the tableau, however, the tableau usually loses most of this sparsity!
  RSM, using the original tableau, can still take advantage of sparsity.

Advantages of Revised Simplex Method

- Because computation is done using only the basis inverse and the original data, the build-up of round-off errors in the computation is largely avoided.

  (LP software will periodically re-invert the basis inverse "from scratch" to avoid this buildup of round-off errors.)

The Revised Simplex Method (or some variation or extension of it) is used in virtually all commonly-available software for linear programming.

Other common extensions:

- *storage of the basis inverse in "product-form" to save both storage space and computation*
- *factoring the basis inverse into triangular factors to make computation of substitution rates and simplex multipliers more efficient.*
- *simple and general upper bounds on variables*

Exercise

original tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
1	1	1	-4	0	0	0	0
0	1	1	2	1	0	0	9
0	1	1	-1	0	1	0	2
0	-1	1	1	0	0	1	4

current tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
1							
0				1/3	0	-2/3	
0				0	1	1	
0				1/3	0	1/3	

Complete the missing elements!

Solution

Solution:

original tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
1	1	1	-4	0	0	0	0
0	1	1	2	1	0	0	9
0	1	1	-1	0	1	0	2
0	-1	1	1	0	0	1	4

current tableau

-Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
1	0	4	0	1	0	2	17
0	1	-1/3	0	1/3	0	-2/3	1/3
0	0	2	0	0	1	1	6
0	0	2/3	1	1/3	0	1/3	13/3

Exercise

original tableau

-Z	X_1	X_2	X_3	X_4	X_5	RHS
1	**a**	-1	2	0	0	0
0	**b**	**c**	**d**	1	0	6
0	-1	3	**e**	0	1	1

current tableau

-Z	X_1	X_2	X_3	X_4	X_5	RHS
1	0	-7	**j**	**k**	**m**	-9
0	**g**	2	-1	$\frac{1}{2}$	0	**f**
0	**h**	**i**	1	$\frac{1}{2}$	1	4

Find the values of "a" through "m"

Solution

Solution

a= 3
b= 2
c= 4
d= −2
e= 2
f= 3
g= 1
h= 0
i= 5
j= 5
k= $-\frac{3}{2}$
m= 0

original tableau

-Z	X_1	X_2	X_3	X_4	X_5	RHS
1	**a**	-1	2	0	0	0
0	**b**	**c**	**d**	1	0	6
0	-1	3	**e**	0	1	1

current tableau

-Z	X_1	X_2	X_3	X_4	X_5	RHS
1	0	-7	**j**	**k**	**m**	-9
0	**g**	2	-1	$\frac{1}{2}$	0	**f**
0	**h**	**i**	1	$\frac{1}{2}$	1	4