Example: Revised Simplex Method

Consider the LP:

Minimize $z = 3x_1 + 2x_2 + 6x_3$

subject to $\begin{cases} 4x_1 + 8x_2 - x_3 \le 5 \\ 7x_1 - 2x_2 + 2x_3 \ge 4 \end{cases}$

$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

Dept of Mechanical & Industrial Engineering
The University of Iowa

By introducing *slack* and *surplus* variables,

the problem is rewritten with ***equality*** constraints as

Minimize cx subject to Ax=b, x ≥ 0

where

c = [3, 2, 6, 0, 0],

b = [5, 4] and

$$A = \begin{bmatrix} 4 & 8 & -1 & 1 & 0 \\ 7 & -2 & 2 & 0 & -1 \end{bmatrix}.$$

Although x_4 (the slack variable in 1st constraint) can be used as a basic variable in the first row,
the choice of a basic variable in 2nd constraint is not obvious,
*requiring solution of a "Phase One" problem with **artificial** variables introduced.*

*Suppose that **"Phase One"** has found the **initial basis***
B ={1,2} for the constraints, i.e., basic variables x_1 and x_2.

We begin the first iteration of the revised simplex method (RSM)

by computing the **basis inverse matrix**:

B={1,2} $\Rightarrow A^B = \begin{bmatrix} 4 & 8 \\ 7 & -2 \end{bmatrix}$

$$\Rightarrow \left(A^B\right)^{-1} = \begin{bmatrix} 0.03125 & 0.125 \\ 0.10937 & -0.0625 \end{bmatrix}$$

Using the basis inverse matrix, we compute the values of the **current basic variables**,

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(A^B\right)^{-1} b$$

$$= \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.65625 \\ 0.296875 \end{bmatrix}$$

Next we compute the **simplex multiplier vector** π, to be used in "pricing" the *nonbasic* columns:

$$\pi = c_B \left(A^B\right)^{-1}$$

$$= \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3125 & 0.25 \end{bmatrix}$$

Use the simplex multiplier vector π to compute the **reduced cost** of the *nonbasic* variables (x_3,x_4, & x_5), starting with x_3:

$$\overline{c}_3 = c_3 - \pi A^3$$

$$= 0 - [0.3125, 0.25] \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= 5.1875 > 0$$

Since this reduced cost is ***POSITIVE***,

increasing x_3 would **increase** the cost.

So x_3 is **rejected** as a pivot variable.

(If we had been maximizing rather than minimizing, of course, then increasing x_3 would benefit the objective!)

We now proceed to the next nonbasic variable, x_4.

Use the simplex multiplier vector π to compute the reduced cost of the *nonbasic variable* x_4:

$$\overline{c}_4 = c_4 - \pi A^4$$

$$= 0 - [0.3125, 0.25] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -0.3125 < 0$$

Increasing x_4 will improve (i.e. ***lower***) the solution, since its reduced cost is **negative**!

Rather than continuing to "price" the remaining nonbasic variables (in this case, only x_5),
we will proceed by entering x_4 into the basis!
For the ***minimum ratio test***, we need the **substitution rates** of x_4 for the basic variables:

$$\alpha = \left(A^B\right)^{-1} A^j$$

$$= \begin{bmatrix} 0.03125 & 0.125 \\ 0.10937 & -0.0625 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.03125 \\ 0.10937 \end{bmatrix}$$

That is, one unit of x_4 will substitute for 0.03125 units of the first basic variable and 0.10937 of the second.

Perform the **minimum ratio test** to determine which variable leaves the basis.

$$\min\left\{\frac{x_B}{\alpha_B} : \alpha_B > 0\right\} = \min\left\{\frac{0.6562}{0.0312}, \frac{0.29687}{0.10937}\right\}$$

$$= \min \{21, \mathbf{2.7143}\} = 2.7143$$

Since the ***second*** ratio is minimum,

the second basic variable (i.e., x_2) is replaced by the entering variable x_4 (which will be 2.7143 in the new basic solution), and

the **new basis** is B = {1, 4}.

Update the basis inverse matrix with a **pivot**:

$$\left[\left(A^B\right)^{-1} | \alpha\right] = \left[\begin{array}{ccc} 0.03125 & 0.125 & 0.03125 \\ 0.10937 & -0.0625 & \boxed{0.109375} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc} 0 & 0.1428 & 0 \\ 1 & -0.5714 & 1 \end{array}\right]$$

$$\Rightarrow \left(A^B\right)^{-1} = \begin{bmatrix} 0 & 0.1428 \\ 1 & -0.5714 \end{bmatrix}$$

For the new basis B={1,4},

$$c_B = [3,0], A^B = \begin{bmatrix} 4 & 1 \\ 7 & 0 \end{bmatrix}, \left(A^B\right)^{-1} = \begin{bmatrix} 0 & 0.142857 \\ 1 & -0.571429 \end{bmatrix}$$

The **basic variables** are *[x_1, x_4]* =

$$x_B = \left(A^B\right)^{-1} b = [0.571429 \quad 2.71429]$$

and the new **simplex multipliers** are

$$\pi = c_B \left(A^B\right)^{-1} = [0 \quad 0.428571]$$

The ***reduced costs*** of the nonbasic variables {2, 3, 5} are now:

$$\bar{c}_2 = c_2 - \pi A^2 = 2 - \begin{bmatrix} 0 & 0.428571 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \end{bmatrix} = 2.85714 > 0$$

$$\bar{c}_3 = c_3 - \pi A^3 = 6 - \begin{bmatrix} 0 & 0.428571 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 5.14286 > 0$$

$$\bar{c}_5 = c_5 - \pi A^5 = 0 - \begin{bmatrix} 0 & 0.428571 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0.428571 > 0$$

Since the reduced costs are all **positive**,

the current solution $\begin{bmatrix} x_1 & x_4 \end{bmatrix} = \begin{bmatrix} 0.571429 & 2.71429 \end{bmatrix}$

is **optimal**!