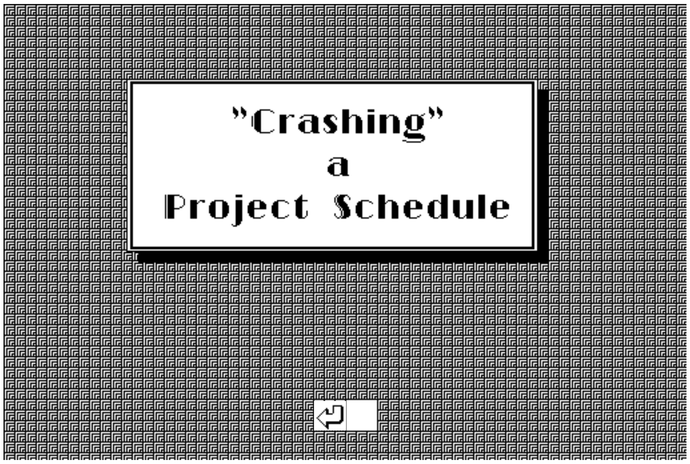


"Crashing" a Project Schedule

-- shortening the duration of a project by shortening the durations of one or more tasks

--additional cost is incurred for assigning extra resources (capital, men, materials, etc.) in order to lower the task durations

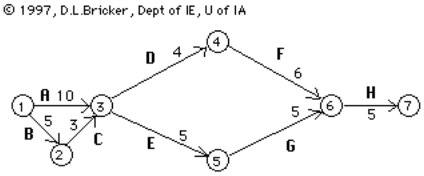
Example Consider a project with 8 activities

Activity	predecessor activities	Normal time (days)	Crash time (days)	Crashing Cost ($/day)
A	--	10	7	4
B	--	5	4	2
C	B	3	2	2
D	A,C	4	3	3
E	A,C	5	3	3
F	D	6	3	5
G	E	5	2	1
H	F,G	5	4	4

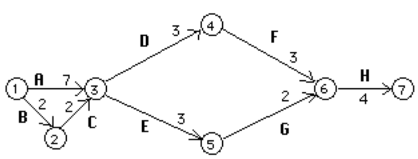
Example

Suppose that overhead costs of the project are $5/day

What is the optimal duration of the project? (i.e., minimizing the sum of **overhead** and **crashing** costs)

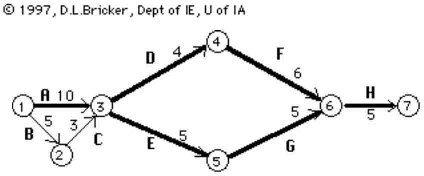


NO-CRASH SCHEDULE
Duration: 25 days
Cost:
overhead $125
crashing $ 0
total $125

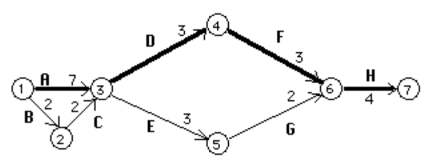


MAX-CRASH SCHEDULE
Duration:17 days
Cost:
overhead $ 85
crashing $ 47
total $132

Is there a lower-cost schedule between these two "extremes"?



NO-CRASH SCHEDULE
Duration: 25 days
Cost:
overhead $125
crashing $ 0
total $125



MAX-CRASH SCHEDULE
Duration:17 days
Cost:
overhead $ 85
crashing $ 47
total $132

Is there a lower-cost schedule between these two "extremes"?

LP Model

Recall the LP formulation of the critical path problem:

decision variables: Y_i = starting time for activity i
constants: d_i = duration of activity i

For the "crashing" problem, the reduction R_i in the duration of activity i is a decision variable as well, with $0 \le R_i \le \overline{d}_i - \underline{d}_i$

where $\overline{d}_i$ = normal duration
$\underline{d}_i$ = minimum duration

The previous precedence constraints were of the form

i is predecessor of j $\Longleftrightarrow$ $Y_j \ge Y_i + d_i$

The new precedence constraints are of the form

i is predecessor of j $\Longleftrightarrow$ $Y_j \ge Y_i + (\overline{d}_i - R_i)$

i.e., $Y_j - Y_i + R_i \ge \overline{d}_i$

where $0 \le R_i \le \overline{d}_i - \underline{d}_i$

```
MIN    4 RA + 2 RB + 2 RC + 3 RD + 3 RE + 5 RF + RG + 4 RH
          + 5 COMPLETE
SUBJECT TO
    2)          YC   - YB  + RB    >=    5
    3)          YD   - YA  + RA    >=   10
    4)          YD   - YC  + RC    >=    3
    5)          YE   - YA  + RA    >=   10
    6)          YE   - YC  + RC    >=    3
    7)          YF   - YD  + RD    >=    4
    8)          YG   - YE  + RE    >=    5
    9)          YH   - YF  + RF    >=    6
   10)          YH   - YG  + RG    >=    5
   11)   COMPLETE - YH  + RH    >=    5
END
```

duration of project (→ COMPLETE)

Example LINDO output

In addition, we specify " Simple Upper Bounds":

$R_i \leq \bar{d}_i - \underline{d}_i$

```
SUB    RA    3.00
SUB    RB    1.00
SUB    RC    1.00
SUB    RD    1.00
SUB    RE    2.00
SUB    RF    3.00
SUB    RG    3.00
SUB    RH    1.00
```

Simple Upper Bounds are treated more efficiently by LINDO than additional rows with these constraints

OBJECTIVE FUNCTION VALUE

1) 121.000000

VARIABLE	VALUE	REDUCED COST
RA	2.0000	0.0000
RB	0.0000	1.0000
RC	0.0000	1.0000
RD	1.0000	-1.0000
RE	0.0000	2.0000
RF	0.0000	1.0000
RG	1.0000	0.0000
RH	1.0000	-1.0000

⋮

Amounts by which activities are to be "crashed"

solution, continued:

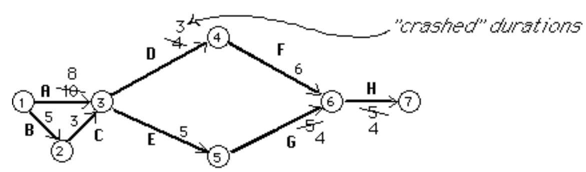
VARIABLE	VALUE	REDUCED COST
YA	0.0000	4.0000
YB	0.0000	1.0000
YC	5.0000	0.0000
YD	8.0000	0.0000
YE	8.0000	0.0000
YF	11.0000	0.0000
YG	13.0000	0.0000
YH	17.0000	0.0000
COMPLETE	21.0000	0.0000

completion times of the activities

solution, continued:

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.0000	-1.0000
3)	0.0000	-4.0000
4)	0.0000	0.0000
5)	0.0000	0.0000
6)	0.0000	-1.0000
7)	0.0000	-4.0000
8)	0.0000	-1.0000
9)	0.0000	-4.0000
10)	0.0000	-1.0000
11)	0.0000	-5.0000

Optimal Solution *duration: 21*



No activity has any slack... all of them lie on a critical path (of which there are four!)

A-D-F-H
A-E-G-H
B-C-D-F-H
B-C-E-G-H

Parametric Programming Approach

By using parametric programming on the right-hand-side, we can plot the project cost (overhead + crashing costs) vs. the project duration.


```
MIN    4 RA + 2 RB + 2 RC + 3 RD + 3 RE + 5 RF + RG + 4 RH
    + 5 COMPLETE
SUBJECT TO
    2)  YC   - YB  + RB  >=   5
    3)  YD   - YA  + RA  >=  10
    4)  YD   - YC  + RC  >=   3
    5)  YE   - YA  + RA  >=  10
    6)  YE   - YC  + RC  >=   3
    7)  YF   - YD  + RD  >=   4
    8)  YG   - YE  + RE  >=   5
    9)  YH   - YF  + RF  >=   6
   10)  YH   - YG  + RG  >=   5
   11)  COMPLETE - YH + RH >=   5
   12)  COMPLETE =   25
END
```

We add row (12), which initially sets project completion time equal to the no-crash time

Simple Upper Bounds on the reductions in activity durations

SUB	RA	3.00
SUB	RB	1.00
SUB	RC	1.00
SUB	RD	1.00
SUB	RE	2.00
SUB	RF	3.00
SUB	RG	3.00
SUB	RH	1.00

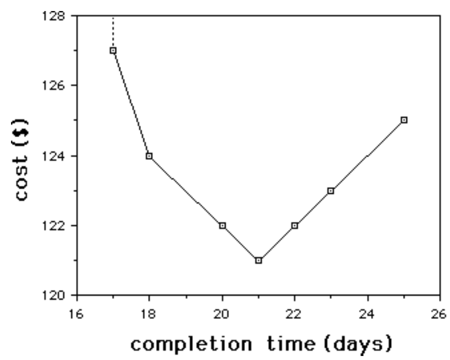
: PARARHS
ROW:12
NEW RHS VAL= 0

We start the parametric analysis, with row (12) right-hand-side to be decreased from 25 to 0 days

VAR OUT	VAR IN	PIVOT ROW	RHS VAL	DUAL PRICE BEFORE PIVOT	OBJ VAL
			25.00	-5.000	125.00
SLK 11	RD	11	25.00	-5.000	125.00
SLK 8	SLK 4	8	25.00	-2.000	125.00
SLK 5	RA	5	25.00	-2.000	125.00
SLK 2	RG	2	23.00	-1.000	123.00
RD	RH	11	22.00	-1.000	122.00
RH	SLK 5	11	21.00	-1.000	121.00
SLK 4	RB	8	21.00	.000	121.00
⋮	⋮	⋮	⋮	⋮	⋮

VAR OUT	VAR IN	PIVOT ROW	RHS VAL	DUAL PRICE BEFORE PIVOT	OBJ VAL
⋮	⋮	⋮	⋮	⋮	⋮
RA	RF	5	20.00	1.000	122.00
RG	SLK 4	2	18.00	1.000	124.00
RB	RC	8	18.00	2.000	124.00
SLK 5	RE	5	18.00	2.000	124.00
RF	ART	9	17.00	3.000	127.00
			.00	+INFINITY	INFEASIBLE

When RHS is reduced to 17, an artificial variable enters the basis. Further reduction is infeasible (would make artificial variable positive!)



Using the output of parametric analysis, we can plot the objective vs. the project duration!

Other Possible Objectives

- Minimize cost of crashing to meet a project deadline T

Minimize $\sum_i C_i R_i$
subject to:
the usual precedence constraints
plus additional constraint:

$$Y_{end} - Y_{begin} \le T$$

Other Possible Objectives

- Minimize project duration, with a given budget B available for crashing costs

Minimize $Y_{end} - Y_{begin}$
subject to:
the usual precedence constraints
plus additional constraint:

$$\sum_i C_i R_i \le B$$

(The "budget" may be resources other than $)